



The Open University

MS324 Waves, diffusion
and variational
principles

0



Block 0 Preparatory material





The Open University

MS324 Waves, diffusion and variational principles

Block 0

Preparatory material

This publication forms part of an Open University course. Details of this and other Open University courses can be obtained from the Student Registration and Enquiry Service, The Open University, PO Box 197, Milton Keynes MK7 6BJ, United Kingdom: tel. +44 (0)845 300 6090, email general-enquiries@open.ac.uk

Alternatively, you may visit the Open University website at <http://www.open.ac.uk> where you can learn more about the wide range of courses and packs offered at all levels by The Open University.

To purchase a selection of Open University course materials visit <http://www.ouw.co.uk>, or contact Open University Worldwide, Walton Hall, Milton Keynes MK7 6AA, United Kingdom, for a brochure: tel. +44 (0)1908 858793, fax +44 (0)1908 858787, email ouw-customer-services@open.ac.uk

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published 2005. Second edition 2006. Third edition 2010.

Copyright © 2005, 2006, 2010 The Open University

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, transmitted or utilised in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without written permission from the publisher or a licence from the Copyright Licensing Agency Ltd. Details of such licences (for reprographic reproduction) may be obtained from the Copyright Licensing Agency Ltd, Saffron House, 6–10 Kirby Street, London EC1N 8TS; website <http://www.cla.co.uk>.

Open University course materials may also be made available in electronic formats for use by students of the University. All rights, including copyright and related rights and database rights, in electronic course materials and their contents are owned by or licensed to The Open University, or otherwise used by The Open University as permitted by applicable law.

In using electronic course materials and their contents you agree that your use will be solely for the purposes of following an Open University course of study or otherwise as licensed by The Open University or its assigns.

Except as permitted above you undertake not to copy, store in any medium (including electronic storage or use in a website), distribute, transmit or retransmit, broadcast, modify or show in public such electronic materials in whole or in part without the prior written consent of The Open University or in accordance with the Copyright, Designs and Patents Act 1988.

Edited, designed and typeset by The Open University, using the Open University TeX System.

Printed in the United Kingdom by Cambrian Printer Limited, Aberystwyth.

ISBN 978 0 7492 5162 8

3.1

The paper used in this publication contains pulp sourced from forests independently certified to the Forest Stewardship Council (FSC) principles and criteria. Chain of custody certification allows the pulp from these forests to be tracked to the end use. (see www.fsc-uk.org).



Contents

CHAPTER 1 MATHEMATICAL METHODS AND DIFFERENTIAL EQUATIONS	5
1.1 Introduction	5
1.2 Some initial definitions and results	5
1.2.1 Functions	5
1.2.2 Series	7
1.2.3 Some elementary functions and their derivatives	10
1.2.4 Integration	12
1.2.5 Partial fractions	18
1.2.6 Complex numbers	21
End-of-section Exercises	27
1.3 Differential equations	28
1.3.1 Initial classification of differential equations	29
1.3.2 Elementary solutions of ordinary differential equations	32
End-of-section Exercises	40
1.4 Outcomes	41
Solutions to Exercises in Chapter 1	42
 CHAPTER 2 MULTIVARIABLE AND VECTOR CALCULUS	 57
2.1 Introduction	57
2.2 Multivariable calculus	58
2.2.1 Level curves	58
2.2.2 Continuity	59
2.2.3 Partial derivatives	61
2.2.4 Linear approximation	63
2.2.5 Local extrema	64
2.2.6 Higher-order partial derivatives	65
2.2.7 Functions of many variables	70
2.2.8 The chain rule	72
2.2.9 Multiple integrals	73
End-of-section Exercises	79
2.3 Vector calculus	80
2.3.1 Vector algebra	80
2.3.2 Scalar fields	82
2.3.3 Vector fields	86
2.3.4 Integration	86
2.3.5 The Gauss divergence theorem	91
2.4 Changing coordinates	96
End-of-section Exercises	100
2.5 Outcomes	101
Solutions to Exercises in Chapter 2	102
 INDEX	 115

CHAPTER I

Mathematical methods and differential equations

1.1 Introduction

In the first two chapters of this course we need to lay the groundwork for the more advanced material that comes later, so here the emphasis is on consolidation of material that you may have seen before. We also need to be sure that you are familiar with the notation that we shall use later.

In this chapter we concentrate on areas that are essential components of an applied mathematician's tool-kit: functions, complex numbers and differential equations. In addition to reminding you of some useful techniques, we also provide you with a number of exercises that are preparatory to material that will be discussed in later chapters.

1.2 Some initial definitions and results

1.2.1 Functions

A **function** f is a mapping from a set A known as the **domain**, to a set B known as the **codomain**, in which every element of A maps to a unique element of B . The notation

$$y = f(x) \quad \text{for } x \in A$$

is used to encapsulate such a relationship, where x is an element of A , y is an element of B , and the symbol $\in$ is an abbreviation for 'is a member of'.

Often A and B are sets of real numbers, in which case f is said to be a **real function**. For example, $f(x) = x^2$ for $x \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, is a real function. The set $\mathbb{R}$ is often referred to as the **real line**. A real function f is **even** if $f(x) = f(-x)$, and **odd** if $f(x) = -f(-x)$. For example, $y = x^2$ is even, while $y = x^3$ is odd.

A function g is an inverse of the function f if $f(g(x)) = x$, and it is common practice to denote an **inverse** of a function f by f^{-1} . For example, if $f(x) = \sqrt{x}$ for $x \geq 0$ (where $\sqrt{x}$ denotes the positive square root of x), then $f^{-1}(x) = x^2$, because $\sqrt{x^2} = x$.

Later we shall meet functions for which the domain is a set of real numbers and the codomain is a set of complex numbers. Such functions are known as **complex functions of a real variable**.

We shall also meet functions for which the domain is itself a set of functions. Such functions are often known as **operators** or **transforms**. This may at first seem a rather strange concept, but you have in fact met the idea before. Whenever you differentiate a function, you are actually applying the differential operator to a function to obtain another function, namely its derivative. So differentiation is in fact an operator that maps functions to their derivatives. We shall not discuss operators in any great depth, but the concept will occur again when we introduce the *Fourier transform* later in the course.

The domains and codomains of specific functions are often defined implicitly. For example, for the real function

$$f(x) = \frac{1}{\sqrt{1-x^2}}$$

it is implicitly assumed that the domain is the set $-1 < x < 1$ since this is the largest set of real numbers on which the definition is sensible. Strictly speaking, we should distinguish between a function f and a function value $f(x)$, but generally we shall make no such distinction and, for example, refer to the function $y = \frac{1}{\sqrt{1-x^2}}$.

Functions themselves are sometimes defined implicitly. For example, the equation (of a circle) $x^2 + y^2 = 1$ determines two functions, $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$, and we require more information in order to discover which of these functions is intended. In either case the domain would be the set $-1 \leq x \leq 1$.

Smooth functions

Often we shall need a function to be 'sufficiently smooth'. The minimum requirement for this is that the function should be **continuous** at the point in question. In intuitive terms, a function is continuous if its graph has no sudden breaks, but we can be a little more precise. A function f is said to be continuous at a point $x = a$ if $\lim_{x \rightarrow a} f(x) = f(a)$. In other words, as x approaches a from above ($x > a$) or below ($x < a$), the function value $f(x)$ approaches the value $f(a)$.

In order for the function to be 'smooth', we also require that it should be **differentiable** at the point in question. This means that for a function f to be smooth at the point $x = a$, we require that the limit $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ exists. Again in intuitive terms, a smooth function is one for which the graph has a tangent at the point in question.

The **derivative** of a function $y = f(x)$ is often denoted by $\frac{dy}{dx}$, although we also denote the **derived function** by f' , so $f'(x) = \frac{dy}{dx}$.

The second derivative is usually written as $f''(x)$ or $\frac{d^2y}{dx^2}$, while the higher derivatives are written as $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$ for $n = 3, 4, \dots$

When we require a function to be 'sufficiently smooth', we generally mean that the function should be differentiable a sufficient number of times at the

It can be shown that a function that is differentiable at a point must be continuous there.

relevant points of its domain. Sometimes this may imply that the function should be differentiable ‘an infinite number of times’. (For example, for the sine function $f(x) = \sin x$, we have $f'(x) = \cos x$ and $f''(x) = -\sin x$, thus the function is differentiable as many times as we please.)

A **polynomial** is a function of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

and the polynomial is of **degree** n if $a_n \neq 0$. Clearly a polynomial is ‘sufficiently smooth’ at any point on the real line because it can be differentiated as many times as we please.

A **rational function** is a function of the form $\frac{p(x)}{q(x)}$ where p and q are polynomials, and such a function is ‘sufficiently smooth’ at every point on the real line except the roots of the equation $q(x) = 0$. At those points the rational function is discontinuous.

1.2.2 Series

Convergence of series

Example 1.1

You may have already met the **geometric series**

$$S_n(a) = 1 + a + a^2 + \cdots + a^{n-1},$$

where n is an integer and a is some specified number. Such a sum can be written in an alternative form if we multiply both sides of the above equation by a to obtain

$$aS_n(a) = a + a^2 + \cdots + a^n,$$

then subtract one equation from the other and rearrange:

$$S_n(a) = \frac{1 - a^n}{1 - a}. \quad (1.1)$$

For large values of n , and for any choice of the number a in the interval $-1 < a < 1$, the term $\frac{a^n}{1 - a}$ is small, therefore $S_n(a) \simeq \frac{1}{1 - a}$. More precisely,

$$\lim_{n \rightarrow \infty} S_n(a) = \frac{1}{1 - a}, \quad (1.2)$$

provided that $-1 < a < 1$. ■

It is this final statement that leads us to the definition of the convergence of an **infinite series**. We begin by considering a finite sum

$$S_n = A_0 + A_1 + A_2 + \cdots + A_{n-1},$$

which may be conveniently written in the form $S_n = \sum_{k=0}^{n-1} A_k$. We then say that the infinite series $\sum_{k=0}^{\infty} A_k$ is **convergent** to the sum S if $\lim_{n \rightarrow \infty} S_n = S$.

If the limit $\lim_{n \rightarrow \infty} S_n$ does not exist, then we say that the series is **divergent**.

A finite sum such as $\sum_{k=0}^{n-1} A_k$ is often known as a **partial sum** of the infinite series.

The symbol $\simeq$ is read as ‘is approximately equal to’.

For the previous example we could then write $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$ if $-1 < a < 1$, and $S_n(a) = 1 + a + a^2 + \cdots + a^{n-1}$ would be the n th partial sum of the infinite series.

Sometimes it is convenient to write an infinite series in a more informal manner, and for example we may write

$$\sum_{k=0}^{\infty} A_k = A_0 + A_1 + A_2 + \cdots + A_n + \cdots,$$

where the final set of dots indicates that the series continues indefinitely.

Exercise 1.1

Show that for $-1 < a < 1$, $R_n(a) = \sum_{k=n}^{\infty} a^k$ can be written in the form

$$R_n(a) = \frac{a^n}{1-a}.$$

Rate of convergence

In some circumstances it is not sufficient simply to know that an infinite series converges; we may also need to know how *rapidly* the series converges. In the previous example we saw that the infinite sum was $\frac{1}{1-a}$ if $-1 < a < 1$, which means that $S_n(a) - \frac{1}{1-a}$ is small when n is large. It is instructive to rearrange equation (1.2) into the form

$$\frac{1}{1-a} = 1 + a + a^2 + \cdots + a^{n-1} + R_n(a),$$

where $R_n(a) = \frac{a^n}{1-a}$ is known as the **remainder term**. The rate at which the remainder term tends to zero determines the **rate of convergence** of the series, and, loosely speaking, in this case we can see that $R_n(a)$ behaves like a^n . However, intuitive statements like this can be made more precise by introducing some additional notation.

The O notation

The **O notation** is a convenient device for specifying an order of magnitude. For example, $\sin x$ is of the same order of magnitude as x when x is small. To make such a statement more precise, we define $f(x) = O(g(x))$ if there exists a constant M such that $|f(x)/g(x)| \leq M$ for all sufficiently small values of x . Thus, for example, it is possible to show that $\sin x = O(x)$ and $2x + 3x^2 = O(x)$ if x is small. In the context of our previous example, we could write $R_n(a) = O(a^n)$ for a sufficiently small, to indicate that $|R_n(a)/a^n| \leq M$ for some constant $M > 1$. So if we are concerned only with orders of magnitude, we could write

$$\frac{1}{1-a} = 1 + a + a^2 + \cdots + a^{n-1} + O(a^n).$$

More generally, for a function $f(x) = g(x) + a_n x^n + a_{n+1} x^{n+1} + \cdots$ with constant coefficients $a_n, a_{n+1}, \dots$, we can write $f(x) = g(x) + O(x^n)$ for sufficiently small values of x .

Example 1.2

Given $f(x) = O(x^2)$ and $g(x) = O(x^3)$ when x is small, establish the orders of magnitude of $f(x)g(x)$ and $f(x) + g(x)$.

Solution

We know that $|f(x)| \leq k_1|x|^2$ and $|g(x)| \leq k_2|x|^3$, for some constants k_1 and k_2 . Therefore $|f(x)g(x)| \leq k_1k_2|x|^5$, so $f(x)g(x) = O(x^5)$.

Also, we have

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq k_1|x|^2 + k_2|x|^3 < 2k_1|x|^2$$

if x is sufficiently small. Hence $f(x) + g(x) = O(x^2)$. ■

Here we have used the triangle inequality

$$|a + b| \leq |a| + |b|$$

(see Exercise 1.14).

Taylor series

It can be shown that a function f that is sufficiently smooth near a point $x = \alpha$ can be represented as a series

$$f(x) = \sum_{k=0}^{\infty} a_k(x - \alpha)^k \quad \text{where } a_k = \frac{f^{(k)}(\alpha)}{k!}. \quad (1.3)$$

Note that

$$0! = 1$$

and

$$f^{(0)}(\alpha) = f(\alpha).$$

Such a series is known as the **Taylor series** (or Taylor expansion) for f near the point $x = \alpha$, and it is generally convergent for x in an interval $\alpha - \rho < x < \alpha + \rho$ (which can be written as $|x - \alpha| < \rho$) for some positive constant ρ . The largest possible choice for ρ is known as the **radius of convergence** of the series.

A sequence of numbers such as $a_1, a_2, a_3, \dots$ is often abbreviated to $\{a_k\}$, and in this case these numbers are known as the **coefficients** of the series.

Example 1.3

Write down the Taylor series for the function $f(x) = \frac{1}{1+x}$ near $x = 0$.

Solution

For this function, $f^{(k)}(x) = \frac{(-1)^k k!}{(1+x)^{k+1}}$. So $f^{(k)}(0) = (-1)^k k!$, and the coefficients of the Taylor series are $a_k = (-1)^k$. Thus

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k. \quad \blacksquare$$

?

Exercise 1.2

Find the largest interval $|x| < \rho$ on which the series

$$1 - x^2 + x^4 - x^6 + \dots = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

is convergent.

1.2.3 Some elementary functions and their derivatives

You will probably have previously encountered most of the following functions and their Taylor series near $x = 0$. (We include the geometric series again for completeness.) We also give the interval of convergence for each series.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{k=0}^{\infty} x^k \quad \text{for } |x| < 1,$$

$$\exp x = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for } x \in \mathbb{R},$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \text{for } x \in \mathbb{R},$$

$$\cos x = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad \text{for } x \in \mathbb{R},$$

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \cdots + \frac{x^n}{n} + \cdots\right) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \quad \text{for } -1 \leq x < 1.$$

For this course, the natural logarithm of x will be denoted by $\ln x$, although other authors write $\log_e x$ or just $\log x$. For $x > 0$ the function $\ln x$ is defined to be the inverse of the exponential function, so $\exp(\ln x) = x$ for $x > 0$. The radius of convergence of the Taylor series for $\ln(1-x)$ near $x = 0$ is 1, but the series diverges at $x = 1$ and converges at $x = -1$.

$\ln x$ is the logarithm to base e .

Many other series can be obtained by simple manipulations of the above series, as the following example shows.

Example 1.4

Write down the Taylor series for the following functions near $t = 0$.

(a) $\ln(1+2t)$ (b) $\frac{1}{(1-3t)^2}$

Solution

(a) Replacing x by $-2t$ in the series for $\ln(1-x)$, we obtain

$$\ln(1+2t) = -\sum_{k=1}^{\infty} \frac{(-2t)^k}{k}.$$

(b) It can be shown that we are justified in differentiating the geometric series term by term to obtain

$$\begin{aligned} \frac{1}{(1-x)^2} &= \frac{d}{dx} \left(\frac{1}{1-x} \right) = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots \\ &= \sum_{k=1}^{\infty} kx^{k-1}. \end{aligned}$$

Then replacing x by $3t$ gives

$$\frac{1}{(1-3t)^2} = 1 + 6t + 27t^2 + \cdots + n(3t)^{n-1} + \cdots = \sum_{k=1}^{\infty} k(3t)^{k-1}. \quad \blacksquare$$

Exercise 1.3

Show that $\sin x \ln(1+x) = x^2 + O(x^3)$.

The remaining trigonometric functions can be defined in terms of sine and cosine:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \operatorname{cosec} x = \frac{1}{\sin x}.$$

The hyperbolic functions

The hyperbolic sine and cosine functions, $\sinh$ and $\cosh$, are defined by

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \text{convergent for } x \in \mathbb{R}, \\ \cosh x &= \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \text{convergent for } x \in \mathbb{R}. \end{aligned}$$

$\sinh$ and $\cosh$ are pronounced 'shine' and 'cosh', respectively.

The remaining hyperbolic functions are defined by

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}, \\ \operatorname{sech} x &= \frac{1}{\cosh x}, \quad \operatorname{cosech} x = \frac{1}{\sinh x}. \end{aligned}$$

$\tanh$, $\coth$, sech and cosech are pronounced 'tanh', 'coth', 'sech' and 'co-sech', respectively.

The graphs of $\sinh$, $\cosh$ and $\tanh$ are shown in Figure 1.1.

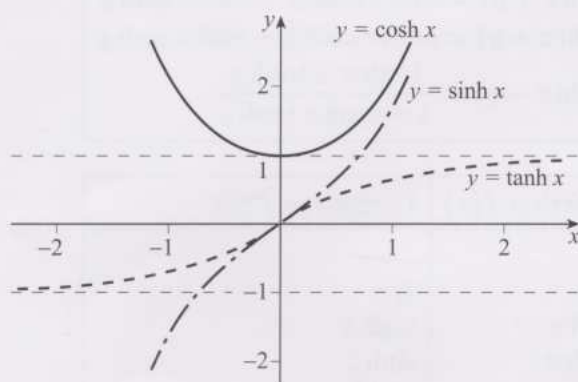


Figure 1.1 The graphs of $\sinh$, $\cosh$ and $\tanh$

Exercise 1.4

From the definitions of $\sinh$ and $\cosh$, show that $\cosh^2 x - \sinh^2 x = 1$.

Example 1.5

Show that $\frac{d}{dx} (\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$.

Solution

If $w = \cosh^{-1} x$, then $\cosh w = x$. Differentiating with respect to x , using the chain rule, we have $\sinh w \frac{dw}{dx} = 1$. Thus

$$\frac{dw}{dx} = \frac{1}{\sinh w} = \frac{1}{\sqrt{\cosh^2 w - 1}} = \frac{1}{\sqrt{x^2 - 1}}. \quad \blacksquare$$

The Gaussian function

The Gaussian function

$y = \exp[-(x - \mu)^2/2\sigma^2],$

where μ and σ are constants, arises in statistics. Its graph is shown in Figure 1.2 for various values of μ and σ .

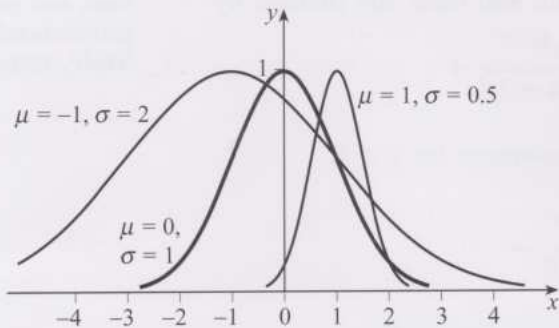


Figure 1.2 Graphs of the Gaussian function

Identities and derivatives

Various identities for the trigonometric and hyperbolic functions arise quite often, and you may find the following lists useful.

$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$\sin(x + y) = \sin x \cos y + \cos x \sin y$	$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
$\cos(x + y) = \cos x \cos y - \sin x \sin y$	$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$	$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

Function $f(x)$	Derivative $f'(x)$	Function $f(x)$	Derivative $f'(x)$
$x^n \quad (n \neq 0)$	nx^{n-1}	$\ln x$	$1/x$
e^x	e^x	$\sinh x$	$\cosh x$
$\sin x$	$\cos x$	$\cosh x$	$\sinh x$
$\cos x$	$-\sin x$	$\tanh x$	$\operatorname{sech}^2 x$
$\tan x$	$\sec^2 x$	$\coth x$	$-\operatorname{cosech}^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$	$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\sec x$	$\sec x \tan x$	$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2 + 1}}$
$\sin^{-1} x$	$\frac{1}{\sqrt{1 - x^2}}$	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2 - 1}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1 - x^2}}$	$\tanh^{-1} x$	$\frac{1}{1 - x^2}$
$\tan^{-1} x$	$\frac{1}{1 + x^2}$		

1.2.4 Integration

A **primitive** of a given function f is a function F such that $F' = f$. The term ‘indefinite integral’ is often used as an alternative to ‘primitive’, and the notation $\int f(x) dx$ is used to indicate an arbitrary indefinite integral of f . Thus indefinite integration arises as the ‘inverse of differentiation’.

On the other hand, **definite integration** originates from the need to evaluate a limit of a sum, for example arising from the area under a curve.

These two notions of integration are linked by the *fundamental theorem of calculus* (see below), but it is important to appreciate that they are quite distinct concepts.

In an integral of the form $\int f(x) dx$, the function $f(x)$ being integrated is called the **integrand**, and the variable x is called the integration variable. Another common way of expressing such integrals is to write the integrand last, i.e. $\int dx f(x)$. Both notations are used in this course.

Definite integration is based on the concept of a limit of a sum. Suppose that we are given a function $f(x)$ defined on the interval $a \leq x \leq b$. We first divide the interval into N subintervals, with end points

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b.$$

Then we form the sum $S = \sum_{r=1}^N f(x_r)(x_r - x_{r-1})$. Often it is convenient to choose subintervals of equal length, which we might denote as δx , and then

this sum becomes $S = \sum_{r=1}^N f(x_r) \delta x$, which is often abbreviated to $\sum f(x) \delta x$.

Finally, we take the limit as the number of intervals increases, and the maximum length of the subintervals shrinks to zero, giving

$$\int_a^b f(x) dx = \lim_{N \rightarrow \infty} \sum_{r=1}^N f(x_r)(x_r - x_{r-1}).$$

It is important to appreciate that this is how the integral is defined, but to *evaluate* an integral we generally try to employ the **fundamental theorem of calculus**. This tells us that if we are able to find a primitive $F(x)$ such that $F'(x) = f(x)$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Example 1.6

Find the area bounded by the curve $y = \ln x$, the x -axis, and the lines $x = 2$ and $x = 3$ (see Figure 1.3).

Solution

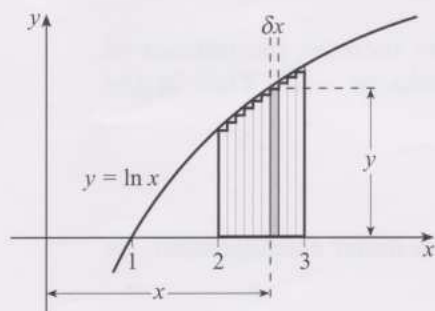


Figure 1.3

If we approximate the required area A by rectangles, as shown in the figure, then $A \simeq \sum \ln(x) \delta x$. Now we take the limit as the number of intervals increases, and the length of the subintervals shrinks to zero, to obtain

$$A = \int_2^3 \ln(x) dx.$$

In an elementary introduction to calculus, such a sum S is usually identified with an estimate for the area under the graph of $f(x)$ from $x = a$ to $x = b$.

To evaluate this integral we may use integration by parts, which gives

$$\begin{aligned} A &= \int_2^3 \ln(x) dx = [x \ln x]_2^3 - \int_2^3 dx \\ &= [x \ln x - x]_2^3 \\ &= 3 \ln 3 - 2 \ln 2 - 1 = \ln(27/4) - 1. \quad \blacksquare \end{aligned}$$

Try differentiating $x \ln x$ if you cannot see why this works.

Example 1.7

Suppose that we are given a wire that does not vary in any way along its length: i.e. its cross-section stays the same, and the material from which it is made remains fixed. We say that such a wire is **uniform**, and we expect the mass of any segment of such a wire to be proportional to the length of the segment: a piece twice as long would weigh twice as much.

The ratio of the total mass of the wire to its total length is known as its **linear density**, and if we denote this quantity by ρ , then the mass M of a length L of wire is ρL . If M is measured in kilograms (kg) and L in metres (m), then the units of ρ are kilograms per metre (kg m^{-1}).

Now suppose that the wire is non-uniform: how should we extend our definition of linear density? Imagine that the wire is placed along the x -axis, and concentrate attention on a segment of wire of length δx containing a particular point x . If the mass of this segment is δm , then we define the linear density $\rho(x)$ at x to be $\lim_{\delta x \rightarrow 0} \frac{\delta m}{\delta x}$.

In order to be more specific, let us consider a wire which is 1 metre long, for which $\rho(x) = 2x + 1$, so that the density increases steadily from the value 1 kg m^{-1} at the origin to 3 kg m^{-1} at the other end of the wire. The total mass of the wire can then be expressed as a definite integral.

First we divide the wire into N subintervals, with end points

$$0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = 1.$$

If the intervals are small, then $\rho(x_r)$ is a good approximation to the linear density at all points of the interval $x_{r-1} \leq x \leq x_r$, so the mass of this short segment of wire is $\delta m_r \simeq \rho(x_r)(x_r - x_{r-1})$. Adding all of these small masses, we obtain an approximation for the total mass of the wire:

$$\begin{aligned} M &= \sum_{r=1}^n \delta m_r \simeq \sum_{r=1}^n \rho(x_r)(x_r - x_{r-1}) \\ &= \sum_{r=1}^n \rho(x_r) \delta x_r, \quad \text{where } \delta x_r = x_r - x_{r-1}. \end{aligned}$$

If we choose equal subintervals, then this approximation can be written as $M \simeq \sum \rho(x) \delta x$.

This approximation becomes more accurate as we increase the number of intervals and the length of each subinterval shrinks to zero. Thus in the limit we have

$$M = \int_0^1 \rho(x) dx.$$

To evaluate this integral for the particular function under consideration, we employ the fundamental theorem of calculus:

$$M = \int_0^1 \rho(x) dx = \int_0^1 (2x + 1) dx = [x^2 + x]_0^1 = 2,$$

i.e. the wire has a mass of 2 kg. $\blacksquare$

Example 1.8

Calculate the volume of a sphere of radius r .

Solution

First divide the sphere into a number of vertical slices of thickness δx (rather like slicing an onion), then approximate a slice at position x by a cylinder of depth δx and radius $y = \sqrt{r^2 - x^2}$ (see the shaded region in Figure 1.4).

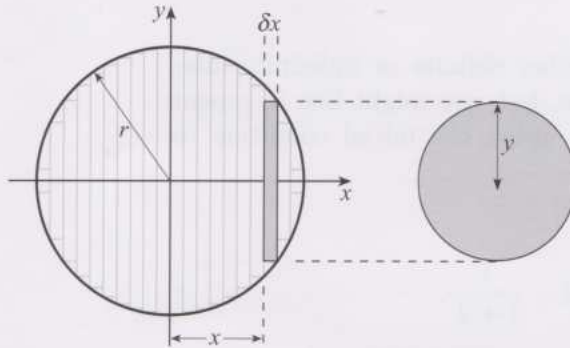


Figure 1.4 A sphere divided into elemental 'cylindrical slices'

The volume of such a cylinder is $\pi y^2 \delta x = \pi(r^2 - x^2) \delta x$, so the sum

$$S = \sum \pi y^2 \delta x = \pi \sum (r^2 - x^2) \delta x$$

provides an approximation to the required volume. In the limit as the number of subdivisions increases and δx tends to zero, the sum becomes a definite

integral and we obtain the required volume, $V = \pi \int_{-r}^r (r^2 - x^2) dx$. Now we may use the fundamental theorem of calculus to obtain

$$V = \pi \int_{-r}^r (r^2 - x^2) dx = \pi \left[r^2 x - \frac{1}{3} x^3 \right]_{-r}^r = \frac{4}{3} \pi r^3. \quad \blacksquare$$

Integrals over an infinite range

The integrals discussed so far have been defined over finite intervals, but we shall also need integrals defined over infinite ranges.

Example 1.9

Suppose that we are told that an object is at the origin at time $t = 0$, and that it moves along the x -axis in such a way that its velocity at time t is given by $v = \cos t$. What is the maximum displacement of the object from the origin?

Solution

We know that $\frac{dx}{dt} = v = \cos t$, so $x(t)$ is a primitive (or indefinite integral) of v . Thus $x(t) = \int \cos t dt = \sin t + c$, where c is a constant. We choose c so that $x(t)$ satisfies the initial condition $x(0) = 0$, thus $c = 0$. So $x(t) = \sin t$, and it is clear that the maximum value of $x(t)$ is 1.

Note that this result can also be found using definite integration, by setting the lower limit equal to the initial condition $x(0) = 0$. At any given time T ,

$$\text{we have } x(T) = \int_0^T \cos t dt = \sin T. \quad \blacksquare$$

In the previous example the object moved backwards and forwards about the origin (consider the graph of $x(t) = \sin t$), but what happens if the object moves always away from the origin but at a slower and slower pace?

Example 1.10

Suppose that we are told that an object is at the origin at time $t = 0$, and that it moves along the x -axis in such a way that its velocity at time t is given by $v = 1/(1+t)^2$. What is the maximum displacement of the object from the origin?

Solution

As in the previous example, we can use either definite or indefinite integration. We choose definite integration here, but you might like to repeat the calculation using indefinite integration, using the initial condition to calculate the value of the arbitrary constant.

At any given time T , we have

$$x(T) = \int_0^T \frac{1}{(1+t)^2} dt = \left[-\frac{1}{1+t} \right]_0^T = 1 - \frac{1}{1+T}.$$

As time T increases, the displacement $x(T)$ increases from zero, but we can see from Figure 1.5 that it never actually achieves a maximum value. Note also that the slope of the graph, $v(T)$, decreases as T increases.

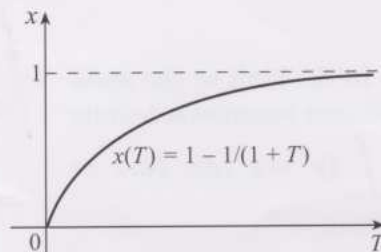


Figure 1.5 The graph of $x(T) = 1 - 1/(1+T)$

As T increases, the value of $x(T) = 1 - \frac{1}{1+T}$ approaches the value 1 from below (but more and more slowly as the speed of the object decreases). So

$$\lim_{T \rightarrow \infty} \left(\int_0^T \frac{1}{(1+t)^2} dt \right) = 1, \text{ and we normally write } \int_0^\infty \frac{1}{(1+t)^2} dt = 1. \quad \blacksquare$$

Thus we are led to our general definition of an integral over an infinite range.

If the limit $\lim_{T \rightarrow \infty} \left(\int_a^T f(t) dt \right)$ exists and equals a finite value L , then we say that the **infinite integral** is **convergent** and write $\int_a^\infty f(t) dt = L$.

If the limit does not exist, then we say that the integral is **divergent**.

In the context of the previous examples, we would not expect $\int_a^\infty v(t) dt$ to exist if the object fails to slow down, but you might reasonably expect that the condition $\lim_{t \rightarrow \infty} v(t) = 0$ would be sufficient to ensure the convergence of $\int_a^\infty v(t) dt$. However, this is not the case, as the following example shows.

Example 1.11

Suppose that we are told that an object is at the origin at time $t = 0$, and that it moves along the x -axis in such a way that its velocity at time t is given by $v = 1/(1+t)$. What is the maximum displacement of the object from the origin?

Solution

As in the previous example, the object moves more and more slowly as it moves away from the origin. However, we now have

$$x(T) = \int_0^T \frac{1}{1+t} dt = \ln(1+T),$$

and $x(T)$ increases without bound as T increases. So here we have a circumstance where the object is moving ever more slowly, but eventually it will move beyond any fixed point that we care to mention. In this case

$$\int_0^\infty \frac{1}{1+t} dt \text{ does not exist, and the integral is divergent.} \blacksquare$$

Sometimes it is possible to establish that an integral is convergent without actually calculating the value of the limit. Suppose that $v(t) \geq 0$ and that

$$x(T) = \int_a^T v(t) dt < K$$

for some fixed constant K and all $T \geq a$. In this case, differentiating with respect to T gives $x'(T) = v(T) \geq 0$, therefore $x(T)$ cannot decrease with time T , but we also know that it cannot exceed the value K . So it must approach some fixed value (which is less than or equal to K).

Example 1.12

Show that the integral $\int_1^\infty \frac{e^{-t}}{t} dt$ is convergent.

Solution

We use the notation introduced in the previous examples. In this case,

$$v(t) = \frac{e^{-t}}{t} > 0 \text{ if } t \geq 1, \text{ so}$$

$$x(T) = \int_1^T \frac{e^{-t}}{t} dt$$

increases with T . But $\frac{e^{-t}}{t} \leq e^{-t}$ if $t > 1$, so

$$x(T) = \int_1^T \frac{e^{-t}}{t} dt \leq \int_1^T e^{-t} dt = [-e^{-t}]_1^T = e^{-1} - e^{-T} < e^{-1}.$$

So $x(T)$ is non-decreasing and bounded above, so we can conclude that

$$\lim_{T \rightarrow \infty} x(T) \text{ exists and therefore the integral } \int_1^\infty \frac{e^{-t}}{t} dt \text{ is convergent.} \blacksquare$$

The Gaussian integral

The following integral identity is very useful in many contexts where Gaussian functions appear:

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2) dx = \sqrt{\frac{\pi}{\alpha}}, \quad \text{where } \alpha > 0. \quad (1.4)$$

This important relation is obtained by a clever trick, where the square of the integral is evaluated using polar coordinates. It will be derived in the next chapter in the section on multiple integrals.

Exercise 1.5

- (a) The following result is often useful when estimating the size of functions that can be expressed as integrals.

If $f(x)$ is a function such that $|f(x)| \leq M$ for $a \leq x \leq b$,

$$\text{then } \left| \int_a^b f(x) dx \right| \leq M(b-a).$$

Use this result to show that $\ln x < x$ for $x \geq 1$.

- (b) Using the method of integration by parts, or otherwise, evaluate the following.

$$(i) \int_1^T \frac{\ln x}{x} dx \quad (ii) \int_1^T \frac{\ln x}{x^2} dx$$

- (c) Show the following.

$$(i) \int_1^\infty \frac{\ln x}{x} dx \text{ is divergent.}$$

$$(ii) \int_1^\infty \frac{\ln x}{x^2} dx \text{ is convergent.}$$

Exercise 1.6

The following result is often useful when comparing the size of integrals.

If $f(x)$ and $g(x)$ are two functions such that $f(x) \leq g(x)$ for $a \leq x \leq b$,

$$\text{then } \int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Use this result to show that $\int_0^\infty \exp(-x^2) dx$ is convergent.

[Hint: First split the integral as

$$\int_0^\infty \exp(-x^2) dx = \int_0^1 \exp(-x^2) dx + \int_1^\infty \exp(-x^2) dx.]$$

1.2.5 Partial fractions

The following process of adding fractions is probably familiar to you:

$$\frac{2}{x-3} + \frac{3}{x+2} = \frac{2(x+2) + 3(x-3)}{(x-3)(x+2)} = \frac{5(x-1)}{(x-3)(x+2)}.$$

The reverse process is known as *expressing a rational function in terms of partial fractions*, and the procedure is useful in differentiation and integration, and when expanding a rational function as a power series.

Suppose that we are told that the identity

$$\frac{5(x-1)}{(x-3)(x+2)} \equiv \frac{A}{x-3} + \frac{B}{x+2} \quad (1.5)$$

holds for some constants A and B , and we want to determine these constants.

There are various methods that can be used to find the values of A and B ; perhaps the simplest is to substitute various well-chosen values of x .

The symbol $\equiv$ is often used in such a case to emphasise the fact that the equality applies for all values of x , but we shall normally use $=$ unless we wish to clarify a point.

For example, putting $x = 0$ in identity (1.5) gives the equation

$$\frac{5}{6} = -\frac{A}{3} + \frac{B}{2},$$

while putting $x = 1$ gives

$$0 = -\frac{A}{2} + \frac{B}{3}.$$

These two equations for A and B are sufficient for us to establish that $A = 2$ and $B = 3$.

Alternatively, we could multiply both sides of identity (1.5) by $(x-3)(x+2)$ to obtain

$$5(x-1) = A(x+2) + B(x-3).$$

Then equating the constant terms and the coefficients of x on both sides of this expression gives

$$-5 = 2A - 3B \quad \text{and} \quad 5 = A + B.$$

Again we have sufficient information to establish that $A = 2$ and $B = 3$. The latter technique is often referred to as *equating coefficients of powers of x* .

This method works for a rational function $\frac{P(x)}{Q(x)}$ where the polynomial $P(x)$ is of lower degree than the polynomial $Q(x)$.

Example 1.13

Express $\frac{1+6x-x^2}{(2x-1)(x+2)(x+1)}$ in partial fractions.

Solution

Since the numerator is of lower degree than the denominator, we attempt to express the fraction in the form

$$\frac{1+6x-x^2}{(2x-1)(x+2)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+2} + \frac{C}{x+1}$$

for some coefficients A , B and C . Then we have

$$1+6x-x^2 = A(x+2)(x+1) + B(2x-1)(x+1) + C(2x-1)(x+2).$$

Now we either set $x = \frac{1}{2}$, $x = -2$, $x = -1$ in turn, or equate coefficients of powers of x , to obtain $A = 1$, $B = -3$, $C = 2$. Thus

$$\frac{1+6x-x^2}{(2x-1)(x+2)(x+1)} = \frac{1}{2x-1} - \frac{3}{x+2} + \frac{2}{x+1}. \quad \blacksquare$$

Exercise 1.7

Express the following in partial fractions.

(a) $\frac{11x+5}{(2x-1)(3x+2)}$

(b) $\frac{2(x+1)(2x+7)}{(x-1)(x+2)(x+3)}$

(c) $\frac{x^3+2x}{x^2-1} \quad \left[\text{Hint: Try writing } \frac{x^3+2x}{x^2-1} = x + \frac{P(x)}{Q(x)}. \right]$

Repeated and quadratic factors

A minor adaptation of the method is required when a factor in the denominator is repeated. For example, to express $\frac{2x-1}{(1-x)^2}$ in partial fractions, we write

$$\frac{2x-1}{(1-x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2}.$$

Multiplying by $(1-x)^2$, we obtain $2x-1 = A(1-x) + B$, and then equating coefficients of powers of x gives $A = -2$ and $B = 1$.

A similar approach will deal with rational functions in which the denominator has a quadratic factor that will not factorise into real factors. For example, to express $\frac{x^2+x+1}{(x^2+1)(x-2)}$ in partial fractions, we write

$$\frac{x^2+x+1}{(x^2+1)(x-2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}.$$

The value of A can be determined by multiplying both sides by $x-2$ and then putting $x = 2$, giving $A = \frac{7}{5}$. Then we multiply by $(x^2+1)(x-2)$ and equate coefficients, giving $B = -\frac{2}{5}$ and $C = \frac{1}{5}$. Thus

$$\frac{x^2+x+1}{(x^2+1)(x-2)} = \frac{1}{5} \left(\frac{7}{x-2} + \frac{1-2x}{x^2+1} \right).$$

All quadratics will factorise if we allow complex numbers in the factors. In fact, the technique discussed above can readily be extended to this case.

Exercise 1.8

Express the following in partial fractions.

- $\frac{x}{x^2-5x+6}$
- $\frac{x^2}{(x-1)^2}$ [Hint: Write $\frac{x^2}{(x-1)^2} = 1 + \frac{P(x)}{Q(x)}$]
- $\frac{(x-1)^2}{(x-3)^2(x-2)}$
- $\frac{x-2}{(2+x^2)(1+x)^2}$

Example 1.14 illustrates some of the applications of partial fractions, but first we remind you of two useful facts.

- $\int \frac{dx}{x+a} = \ln(x+a) + c$ for $x+a > 0$.
- $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$ for $|x| < 1$.

Example 1.14

Use the expression obtained in Example 1.13 to do the following.

- Evaluate the integral $\int \frac{1+6x-x^2}{(2x-1)(x+2)(x+1)} dx$ for $x > \frac{1}{2}$.
- Express $f(x) = \frac{1+6x-x^2}{(2x-1)(x+2)(x+1)}$ as a Taylor series near $x = 0$.
- Calculate $\frac{d^3 f}{dx^3}$.

Solution

(a) Using the fact that $\int \frac{1}{x} dx = \ln x$ for $x > 0$, we see that

$$\begin{aligned} \int \frac{1+6x-x^2}{(2x-1)(x+2)(x+1)} dx \\ &= \int \frac{1}{2x-1} dx - \int \frac{3}{x+2} dx + \int \frac{2}{x+1} dx \\ &= \frac{1}{2} \ln(2x-1) - 3 \ln(x+2) + 2 \ln(x+1) + c, \end{aligned}$$

since $x > \frac{1}{2}$.

(b) First we have

$$\frac{1+6x-x^2}{(2x-1)(x+2)(x+1)} = -\frac{1}{1-2x} - \frac{\frac{3}{2}}{1+\frac{1}{2}x} + \frac{2}{1+x},$$

then we recall that the geometric series can be written as

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \quad \text{for } |x| < 1.$$

Replacing x in this series first by $-\frac{1}{2}x$, then by $2x$, and finally by $-x$, we obtain

$$\begin{aligned} \frac{1+6x-x^2}{(2x-1)(x+2)(x+1)} &= -\frac{3}{2} \left(1 - \frac{1}{2}x + \left(\frac{1}{2}x\right)^2 - \left(\frac{1}{2}x\right)^3 + \cdots \right) \\ &\quad + 2(1 - x + x^2 - x^3 + \cdots) \\ &\quad - (1 + 2x + (2x)^2 + (2x)^3 + \cdots) \end{aligned}$$

for $|x| < \frac{1}{2}$.

(c) First we notice that if $g(x) = (1+ax)^{-1}$, then $\frac{d^3g}{dx^3} = -3!a^3(1+ax)^{-4}$.

So, choosing appropriate values for a , we obtain

$$\begin{aligned} \frac{d^3f}{dx^3} &= -3! \left(\frac{2^3}{(1-2x)^4} - \frac{1}{2^4} \frac{3}{(1+\frac{1}{2}x)^4} + \frac{2}{(1+x)^4} \right) \\ &= -\frac{48}{(1-2x)^4} + \frac{18}{(2+x)^4} - \frac{12}{(1+x)^4}. \quad \blacksquare \end{aligned}$$

1.2.6 Complex numbers

A brief history

Historically, complex numbers arose from attempts to solve equations, such as $x^2 = -1$, which have no real solutions. Early mathematicians would have said that such equations have no solutions, but in the mid-sixteenth century, Cardano and his contemporaries were performing calculations involving $\sqrt{-1}$. They discovered that assuming the existence of this so-called ‘imaginary number’ enabled them to simplify the algebra involved in establishing results involving ‘normal’ numbers. Euler introduced the notation $i = \sqrt{-1}$ in 1777, and established the relation $e^{i\pi} = -1$ (which involves four of the most interesting numbers in mathematics). In 1799 Gauss proved his fundamental theorem of algebra, which states that every polynomial equation has real or complex roots, and early in the nineteenth century Cauchy and others extended the notions of calculus to complex functions. However, these are generally matters that are better discussed in a course on pure mathematics. Here we are mainly concerned with the *applications* of complex numbers; we shall meet them again when we discuss Fourier series and transforms.

Complex algebra

For our purposes it is adequate to introduce the **imaginary number** i with the property that $i^2 = -1$, and to regard the set of complex numbers $\mathbb{C}$ as the set of all 'numbers' $x + iy$, where x and y are real, with the normal rules of algebra. Thus, for example,

$$\begin{aligned}(1 + 2i)^3 &= (1 + 2i)(1 + 2i)(1 + 2i) \\ &= 1 + 3(2i) + 3(2i)^2 + (2i)^3 \\ &= 1 + 6i - 12 - 8i \\ &= -11 - 2i.\end{aligned}$$

In a more formal definition, of the kind that one may find in a course in pure mathematics, the set $\mathbb{C}$ may be defined to be the set of ordered pairs of real numbers (x, y) , with certain specified rules of addition and multiplication. The complex number i is then identified with the ordered pair $(0, 1)$.

Real and imaginary parts

It is common practice to write $z = x + iy$ and then refer to the complex number z , with x and y as its **real** and **imaginary** parts, respectively. We use the form $z = x + iy$ so frequently for an arbitrary complex number that it is not usually necessary to add that we are assuming that x and y are real.

An expression such as $x + iy$ is often known as the **Cartesian form** of a complex number z (by analogy with the Cartesian coordinates of the plane). It is also useful to introduce the notation $\operatorname{Re}(z) = x$ and $\operatorname{Im}(z) = y$ for the real and imaginary parts of the complex number z . You should note particularly that the imaginary part of $z = x + iy$ is y , *not* iy . You should also be aware that two complex numbers can be equal only if both their real and imaginary parts are equal. Thus, for example, if $x + iy = 2 + i\sqrt{3}$, then we know that $x = 2$ and $y = \sqrt{3}$. This process is sometimes known as **equating real and imaginary parts**.

We also identify the set of real numbers with the set of complex numbers having zero imaginary parts. In other words, we regard the real number 3 as being identical to the complex number $3 + 0i$. So a real number may be regarded as a special case of a complex number.

The Argand diagram

Real numbers are often represented as points on a line, whereas complex numbers, having two independent components, require two dimensions. It is for this reason that they are often represented as points in a two-dimensional Cartesian system, and this representation of the complex plane is usually known as the **Argand diagram** (see Figure 1.6). In this representation the x -axis is often known as the real line and represents the set of real numbers. We may also refer to the x - and y -axes as the real and imaginary axes, respectively.

If $z = x + iy$, then the **conjugate** of z is defined to be $z^* = x - iy$ (see Figure 1.6).

Some authors use $\bar{z}$ rather than z^* .

The **modulus** of $z = x + iy$ is defined to be $|z| = \sqrt{x^2 + y^2}$, and the identity $|z|^2 = z z^*$ is often useful (as you will see in the exercises).

It is easy to establish the results

$$|z^*| = |z| \quad \text{and} \quad (z^*)^* = z$$

for any complex number z . Also,

$$|z_1 z_2| = |z_1| |z_2|, \quad |z_1/z_2| = |z_1|/|z_2| \quad \text{and} \quad (z_1 z_2)^* = z_1^* z_2^*$$

for arbitrary complex numbers z_1 and z_2 .

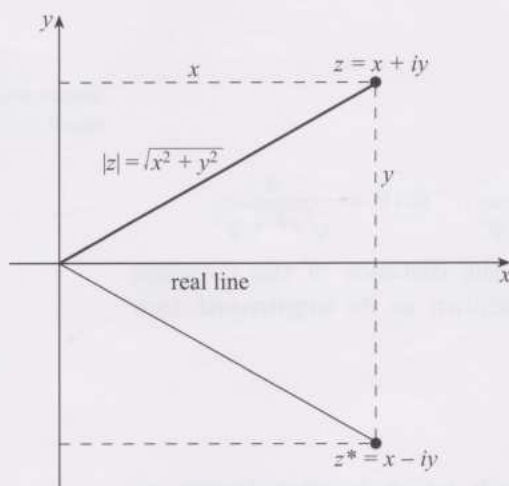


Figure 1.6 The Argand diagram

Example 1.15

Given $z = \frac{2+3i}{1-i}$, find z^* in Cartesian form.

Solution

We can convert z into Cartesian form by writing

$$z = \frac{2+3i}{1-i} \times \frac{1+i}{1+i} = \frac{1}{2}(5i-1);$$

then we can see that

$$z^* = -\frac{1}{2}(1+5i).$$

Alternatively, we can replace every i by $-i$ in the expression for z , to obtain

$$z^* = \frac{2-3i}{1+i}$$

and then

$$z^* = \frac{2-3i}{1+i} \times \frac{1-i}{1-i} = -\frac{1}{2}(1+5i).$$

It would be equally acceptable to write

$$-\left(\frac{5}{2}i + \frac{1}{2}\right) \quad \text{or} \quad -\frac{1}{2} - \frac{5i}{2}. \quad \blacksquare$$

Exercise 1.9

Express $z = \frac{a+bi}{c+di}$ in Cartesian form (where a, b, c and d are real, with $c^2 + d^2 \neq 0$).

Exercise 1.10

Rewrite the complex number $z = (1+3i)^2 - (3-i)^2$ in Cartesian form $a+ib$, where a and b are real.

Find the modulus and the imaginary part of z . Also find z^* , and calculate zz^* .

Polar form and de Moivre's theorem

Using polar coordinates, $z = x + iy$ becomes

$$z = r(\cos \theta + i \sin \theta),$$

known as the **polar form** of z , where

$$r = \sqrt{x^2 + y^2} = |z| \geq 0, \quad \cos \theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin \theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

On an Argand diagram, the **modulus** r is the distance of the complex number from the origin, and the angle θ is known as its **argument** (see Figure 1.7).

It can be shown that

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

for any positive or negative integer n , a result which is often known as **de Moivre's theorem**.

Note that it is possible to define powers of complex numbers z^n for any real number n . In fact, de Moivre's theorem also holds in this case, as will be demonstrated in Exercise 1.13.

Some authors abbreviate $\cos \theta + i \sin \theta$ to $\text{cis } \theta$.

Abraham de Moivre (or Demoivre) was an anglicised Frenchman and a contemporary of Sir Isaac Newton. He adopted various spellings of his name while he lived in England.

Exercise 1.11

Use de Moivre's theorem and the fact that $\cos^2 \theta + \sin^2 \theta = 1$ to show that

$$\cos 5\theta = \cos \theta(16 \cos^4 \theta - 20 \cos^2 \theta + 5).$$

The argument of a complex number

The polar form of a complex number is not unique, because its argument is not unique. For example, the complex number $1 + i$ can be written in polar form as $\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$, but it would be equally valid to write $\sqrt{2}(\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4})$, or indeed $\sqrt{2}(\cos(\frac{\pi}{4} + 2k\pi) + i \sin(\frac{\pi}{4} + 2k\pi))$ for any integer k .

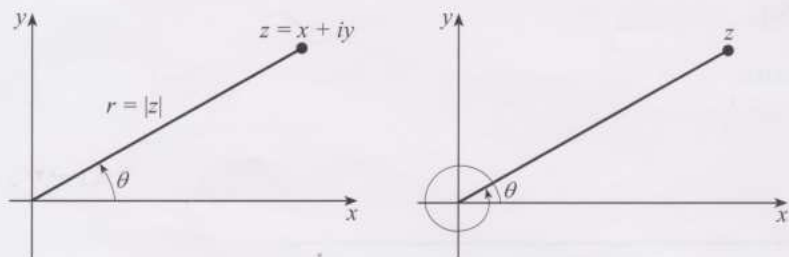


Figure 1.7 Two possible values of the argument of a complex number

An argument of an arbitrary complex number $z = x + iy$ is defined to be an angle θ for which $\sin \theta = y/\sqrt{x^2 + y^2}$ and $\cos \theta = x/\sqrt{x^2 + y^2}$.

If $z = r(\cos \theta + i \sin \theta)$ and $-\pi < \theta \leq \pi$, then θ is known as the **principal value** of the argument of z . An argument of z is written as $\arg(z)$, while the principal value of the argument is written as $\text{Arg}(z)$.

Some authors define the principal value to lie in the interval $0 \leq \theta < 2\pi$.

Example 1.16

Express the complex number $z = \sqrt{3} + i$ in polar form.

Solution

If θ represents an argument of z , then $\cos \theta = \sqrt{3}/2$ and $\sin \theta = 1/2$, and we see that $\frac{\pi}{6}$ is a suitable value for θ . Also, $|z| = 2$, so

$$\sqrt{3} + i = 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right).$$

Any value of the argument of the form $\frac{\pi}{6} + 2k\pi$, where k is an integer, is equally acceptable, but $\frac{\pi}{6}$ is the principal value of the argument. (When determining the argument of a complex number, it is often helpful to plot the point on an Argand diagram.) ■

We can use de Moivre's theorem to find roots of complex numbers. For example, to find the fourth roots of $z = \sqrt{3} + i$, first express it in polar form,

$$z = 2 \left(\cos \left(\frac{\pi}{6} + 2\pi k \right) + i \sin \left(\frac{\pi}{6} + 2\pi k \right) \right),$$

using the general value of the argument, i.e. with k as any integer. Then use de Moivre's theorem to evaluate the fourth roots:

$$z^{1/4} = 2^{1/4} \left(\cos \left(\frac{1}{4} \left(\frac{\pi}{6} + 2\pi k \right) \right) + i \sin \left(\frac{1}{4} \left(\frac{\pi}{6} + 2\pi k \right) \right) \right).$$

For $k = 0, 1, 2, 3$, this gives four distinct complex numbers:

$$\begin{aligned} z_0 &= 2^{1/4} \left(\cos \frac{\pi}{24} + i \sin \frac{\pi}{24} \right), & z_1 &= 2^{1/4} \left(\cos \frac{13\pi}{24} + i \sin \frac{13\pi}{24} \right), \\ z_2 &= 2^{1/4} \left(\cos \frac{25\pi}{24} + i \sin \frac{25\pi}{24} \right), & z_3 &= 2^{1/4} \left(\cos \frac{37\pi}{24} + i \sin \frac{37\pi}{24} \right), \end{aligned}$$

which are the four fourth roots of z . Other values of k simply duplicate these roots – which can, of course, be expressed in Cartesian form if required.

You should know that a complex number has at most n distinct n th roots.

Exercise 1.12

- Express the complex number $z = 2 - 2i$ in polar form.
- Find the cube roots of $z = 2 - 2i$ in Cartesian form.

The complex exponential function

It is possible to extend the notion of Taylor series to complex numbers, and to define

$$\begin{aligned} \exp(i\theta) &= e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \cdots + \frac{(i\theta)^n}{n!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \right) + i \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \right) \\ &= \cos \theta + i \sin \theta. \end{aligned}$$

This result is known as **Euler's formula**.

Any complex number in the form $z = r(\cos \theta + i \sin \theta)$ can then be written in the equivalent form $z = re^{i\theta}$.

We can also use Euler's formula to show that

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

(which reinforces the close relationship between the trigonometric and hyperbolic functions).

Exercise 1.13

Use Euler's formula to prove de Moivre's theorem.

Example 1.17

Express the following numbers in polar form.

(a) $1 + i$ (b) $1 + i\sqrt{3}$ (c) $\frac{1 + i\sqrt{3}}{1 + i}$

Solution

(a) $1 + i = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$

(b) $1 + i\sqrt{3} = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$

(c) Since (using the results of parts (a) and (b)) $1 + i\sqrt{3} = 2e^{i\pi/3}$ and $1 + i = \sqrt{2}e^{i\pi/4}$, it follows that

$$\frac{1 + i\sqrt{3}}{1 + i} = \frac{2e^{i\pi/3}}{\sqrt{2}e^{i\pi/4}} = \sqrt{2}e^{i\pi/12} = \sqrt{2} \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right). \quad \blacksquare$$

In each case a multiple of 2π could be added to the argument.

Complex numbers are often useful in calculations that would otherwise be rather tedious. Suppose, for example, that we wish to evaluate

$$\int e^{at} \sin(bt) dt,$$

where a and b are real. This integral can certainly be found using the method of integration by parts (twice), but the following approach is probably easier.

Consider the integral

$$\begin{aligned} (A) \quad I &= \int e^{at} e^{ibt} dt \\ &= \int e^{(a+ib)t} dt \\ &= \frac{1}{a+ib} e^{(a+ib)t} + C \\ &= e^{at} \frac{\cos(bt) + i \sin(bt)}{a+ib} + C \\ &= e^{at} \frac{\cos(bt) + i \sin(bt)}{a+ib} \times \frac{a-ib}{a-ib} + C \\ &= \frac{e^{at}}{a^2+b^2} ((a \cos(bt) + b \sin(bt)) + i(a \sin(bt) - b \cos(bt))) + C. \end{aligned}$$

$e^{i\alpha} = \cos \alpha + i \sin \alpha$
 $\therefore e^{ibt} = \cos(bt) + i \sin(bt)$

$\frac{1}{a+ib} \cdot e^{at} \cdot e^{ibt}$
 $\frac{1}{a+ib} \cdot e^{at} (\cos(bt) + i \sin(bt))$

We can also write

$$(A) \quad I = \int e^{at} e^{ibt} dt = \int e^{at} (\cos(bt) + i \sin(bt)) dt = \int e^{at} \cos(bt) dt + i \int e^{at} \sin(bt) dt$$

so, equating real and imaginary parts of these alternative representations of I , we obtain

$$\int e^{at} \cos(bt) dt = \frac{e^{at}}{a^2+b^2} (a \cos(bt) + b \sin(bt)) + C_1$$

and

$$\int e^{at} \sin(bt) dt = \frac{e^{at}}{a^2+b^2} (a \sin(bt) - b \cos(bt)) + C_2$$

(where $C = C_1 + iC_2$).

So we have found not only the required integral, but also $\int e^{at} \cos(bt) dt$ with no extra effort.

In a later chapter we shall meet complex numbers again in the context of Fourier series, and the following observations will prove useful.

These become
standard
integrals

For any non-zero integer k ,

$$\int_{-\pi}^{\pi} e^{ik\theta} d\theta = \frac{1}{ik} \left[e^{ik\theta} \right]_{-\pi}^{\pi} = \frac{1}{ik} (e^{ik\pi} - e^{-ik\pi}) = \frac{2}{k} \sin(k\pi) = 0, \quad (1.6)$$

while for $k = 0$ we have

$$\int_{-\pi}^{\pi} e^{ik\theta} d\theta = \int_{-\pi}^{\pi} d\theta = 2\pi. \quad (1.7)$$

To give you some impression of how these results will be used later, consider the function

$$f(\theta) = A_1 e^{i\theta} + A_2 e^{2i\theta} + A_3 e^{3i\theta} + A_4 e^{4i\theta}.$$

With the aid of the stated results, it is very easy to express the coefficients in this expression in terms of the function f . For example, to determine A_3 , multiply by $e^{-3i\theta}$ and integrate from $-\pi$ to π , to obtain

$$\begin{aligned} \int_{-\pi}^{\pi} f(\theta) e^{-3i\theta} d\theta &= A_1 \int_{-\pi}^{\pi} e^{-2i\theta} d\theta + A_2 \int_{-\pi}^{\pi} e^{-i\theta} d\theta \\ &\quad + A_3 \int_{-\pi}^{\pi} d\theta + A_4 \int_{-\pi}^{\pi} e^{i\theta} d\theta, \end{aligned}$$

which gives $A_3 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-3i\theta} d\theta$. In general,

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ni\theta} d\theta$$

for $n = 1, 2, 3, 4$.

End-of-section Exercises

Exercise 1.14

Prove the **triangle inequality**, that $|a + b| \leq |a| + |b|$ for any two real numbers a and b .

Exercise 1.15

Show that

$$\frac{d}{dx}(\sinh x) = \cosh x \quad \text{and} \quad \frac{d}{dx}(\cosh x) = \sinh x.$$

Hence show that $\phi = A \sinh \omega x + B \cosh \omega x$, where A and B are arbitrary constants, is a solution of the differential equation $\frac{d^2 \phi}{dx^2} = \omega^2 \phi$.

Exercise 1.16

Express $\frac{1}{n(n+1)}$ in partial fractions, and hence show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Exercise 1.17

Find the Taylor series for $\sinh(2x)$ near $x = 0$.

Exercise 1.18

Show that $\frac{\sin x}{1-x^2} = x + \frac{5}{6}x^3 + O(x^5)$.

Exercise 1.19

Show that $f(x) = \frac{x}{1+x^2}$ is odd. Hence evaluate $\int_{-1}^1 \frac{x}{1+x^2} dx$.

Exercise 1.20

Show that $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$, and sketch the graph of $y = \sinh^{-1} x$.

Show that $x + \sqrt{1+x^2} > 0$ if $x \in \mathbb{R}$.

Calculate $\frac{d}{dx}(\ln(x + \sqrt{1+x^2}))$, and hence show that $\sinh^{-1} x = \ln(x + \sqrt{1+x^2})$.

Exercise 1.21

Express $\frac{4x^2 - 9x + 4}{x(x-1)(x-2)}$ in partial fractions, and hence evaluate $\int_3^4 \frac{4x^2 - 9x + 4}{x(x-1)(x-2)} dx$.

Exercise 1.22

For what values of the constant a is the integral $\int_1^\infty \frac{1}{t^a} dt$ convergent?

Exercise 1.23

Show that if $\int_a^\infty |f(t)| dt$ is convergent, then $\int_a^\infty f(t) dt$ is also convergent.

[Hint: Consider $f(t) + |f(t)|$.]

Show that $\int_a^\infty \frac{\sin t}{t^2} dt$ is convergent.

This exercise is optional.

Applied mathematics is dependent on many results from pure mathematics. This exercise is intended to give you some indication of the steps that would be required to justify some of our assumed results.

Exercise 1.24

Express the following in Cartesian form.

(a) $\frac{1+3i}{1-2i}$ (b) $\frac{2-3i}{(1-i)^2}$ (c) $\frac{1}{1-i} + \frac{1}{1+i}$

Exercise 1.25

Express the following in polar form.

(a) $\frac{1}{1-i}$ (b) $(1+i\sqrt{3})^2$ (c) $(1+i\sqrt{3})^{15}$

Exercise 1.26

Show that $\cosh(i\theta) = \cos \theta$ and $\sinh(i\theta) = i \sin \theta$.

Exercise 1.27

Show that for any two complex numbers z and w ,

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2.$$

1.3 Differential equations

A brief history

The study of differential equations has a venerable history, and the subject is still central to modern mathematics because many physical systems can be described by differential equations. As early as 1676, Newton solved a differential equation by means of an infinite series, although his results were not published until 1693, the year in which a differential equation appeared for the first time in the work of Leibniz. Shortly afterwards, Johann Bernoulli developed the method of 'separating the variables', and he also showed how

a differential equation of the form $dy/dx = f(y/x)$ could be reduced to one in which the variables were separable. He and his brother Jakob succeeded in reducing a large number of differential equations to soluble forms.

Developments in the eighteenth and nineteenth centuries were rapid and involved some of the greatest names in mathematics: Euler, d'Alembert, Lagrange, Laplace, Cauchy, Jacobi, Ampère, Frobenius and Lie, to mention but a few.

From these passing references to its history, it should be evident that the study of differential equations is a vast subject, and here we can do little more than scratch the surface.

Introduction

While it is important to be able to solve various differential equations, it is of equal importance to be able to recognise that a given equation is of a certain type, for then you can refer to the method of solution in one of the many standard texts. You may have seen some of these differential equations before in other courses, in which case you can regard the corresponding exercises as revision, but they are included here for completeness.

While the variables, constants and parameters in this section are generally real, much of this material applies also when they are complex. However, certain complications arise in integration of complex functions, so for the moment you may assume that everything is real.

1.3.1 Initial classification of differential equations

Essentially, the study of differential equations is a matter of assigning them to various categories and discovering as much as possible about each such category. Here we shall make a start by dividing them into two major groups: **ordinary differential equations** and **partial differential equations**.

Ordinary differential equations arise when we are considering the variation of a physical quantity that depends on just one independent variable. If we denote the independent and dependent variables by x and y , respectively, then an ordinary differential equation may involve functions of x and y , and the derivatives of y with respect to x . Thus

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0, \quad \frac{dy}{dx} + cy = 0 \quad (\text{where } c \text{ is a constant})$$

$$\text{and} \quad \frac{d^2 y}{dx^2} = -k^2 \sin y \quad (\text{where } k \text{ is a constant})$$

are all examples of ordinary differential equations. (Quantities such as c and k in these illustrations are sometimes known as parameters; while the solution may depend on their values, they are generally assumed to be fixed.) Sometimes we omit the word 'ordinary' when it is clear that we are referring to an ordinary differential equation.

On the other hand, partial differential equations arise when the dependent variable is a function of two or more independent variables. If we denote the independent variables by x and t , and the dependent variable by y , then a partial differential equation may involve partial derivatives of y with respect to x and t , as well as functions of x and t ; for example, $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ (where c is a constant). We shall be concerned with such equations and their solutions later in the course.

Here we are primarily concerned with the process of finding solutions, not with the difficulties that may arise in the definitions of the functions and their domains. We assume throughout that unless otherwise stated, the functions involved are well behaved: for example, we assume that they are differentiable as many times as we please on an interval of the real line. (You may take this to mean that the functions are 'sufficiently smooth'.)

Solution of a differential equation

A function that satisfies a given differential equation is known as a *solution* of the differential equation, and, while it may not be obvious how one might find such a function, it is easy to verify that a given suggestion is correct. For example, we can easily verify that the function $y = 3x^2 + Ae^x$ satisfies the differential equation $\frac{dy}{dx} - y = 6x - 3x^2$ for any value of the constant A , by writing

$$\frac{dy}{dx} - y = (6x + Ae^x) - (3x^2 + Ae^x) = 6x - 3x^2.$$

A solution may be required to satisfy certain additional constraints, for example that $y(0) = 0$, and such conditions will further restrict the choice of a suitable function.

It is not always possible to find the required function explicitly; we may have to be content with an implicit definition. For example, it is easy to verify that if $y = y(x)$ is a function for which $x^2 + xy + y^5 = c$, then differentiating with respect to x gives

$$2x + y + x\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0.$$

It follows that $\frac{dy}{dx} = -\frac{y+2x}{x+5y^4}$, so $y = y(x)$ is a solution of this differential equation; but we cannot find an explicit formula for y in terms of x . We may also refer to an implicit definition, such as $x^2 + xy + y^5 = c$, as a solution of a differential equation.

Example 1.18

Show that the function $y(x) = 3\sin(\lambda x)$ is a solution of the differential equation $\frac{d^2y}{dx^2} = -\lambda^2 y$, where λ is a constant.

Solution

In order to establish that this is the case, we need only differentiate y twice, to obtain first $\frac{dy}{dx} = 3\lambda \cos(\lambda x)$ and then

$$\frac{d^2y}{dx^2} = -3\lambda^2 \sin(\lambda x) = -\lambda^2 y,$$

as required. ■

In the previous example the solution was given as a function, but more often such solutions arise implicitly, as in the following case.

Example 1.19

Show that $xy = e^x + e^y$ is a solution of the differential equation $\frac{dy}{dx} = \frac{y - e^x}{e^y - x}$.

Solution

Differentiating the equation $xy = e^x + e^y$ with respect to x , we have $y + xy' = e^x + e^y y'$. Rearranging this equation gives $y'(e^y - x) = y - e^x$, which establishes the required result.

Notice that in this case we are not able to rearrange the original equation into a simple algebraic expression for y in terms of x . ■

Order of a differential equation

The **order** of a differential equation is the order of the highest derivative that appears in it. For example, the differential equation $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y = 0$ is of order two.

From the chain rule, the derivative of $e^{y(x)}$ with respect to x is $y'e^y$.

With the aid of a computer program, it is possible to convince oneself that the equation $xy = e^x + e^y$ defines a function $y = y(x)$ with the negative real axis as its domain.

Exercise 1.28

- (a) In each case, form a differential equation of first order by eliminating the constant a .
- (i) $y = e^{ax}$ (ii) $y = \sin(x + a)$
- (b) Form a differential equation of second order by eliminating the constants a and b from $y = \frac{x+a}{x+b}$.
- (c) Solve the differential equation $\frac{d^2 y}{dx^2} = 1$.
- (d) Show that a function $y = y(x)$ that satisfies the equation $xy + \cos y = 3$ is a solution of the differential equation $\frac{dy}{dx} = \frac{y}{\sin y - x}$.

Part (c) illustrates the fact that we might expect the solution of a second-order equation to contain two arbitrary constants.

Linear differential equations

A **linear ordinary differential equation of order n** is an equation that can be written in the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = f(x), \quad (1.8)$$

with $a_n(x) \neq 0$, for some function $f(x)$.

A solution of a linear differential equation of order n containing n independent constants is known as the **general solution** of the equation. For linear equations it can be shown that every solution can be obtained by assigning values to the independent constants in the general solution. This is not the case for non-linear differential equations.

The differential equation (1.8) is said to be **homogeneous** if $f(x) = 0$, and such an equation has n independent solutions $y_1(x), y_2(x), \dots, y_n(x)$. Moreover, every solution $y(x)$ of the differential equation may be written in the form

$$y(x) = A_1 y_1(x) + A_2 y_2(x) + \cdots + A_n y_n(x), \quad (1.9)$$

where $A_1, A_2, \dots, A_n$ are constants.

In order to find the general solution of equation (1.8) when $f(x) \neq 0$, we need only find a single solution without arbitrary constants (known as a **particular integral**), then add a function of the form (1.9).

In order to solve a practical problem, it is not usually sufficient to determine the general solution of a differential equation; often we need to assign specific values to the constants that arise in the general solution. Such values are generally determined by certain additional restrictions; loosely speaking, one would expect that n pieces of information would be needed to uniquely define the solution of a differential equation of order n .

As a simple illustration, $y = Ae^x + Be^{-x}$ is the general solution of the second-order differential equation $y'' = y$. The conditions $y(0) = 1$ and $y'(0) = 0$ are sufficient to determine the two constants A and B (in fact $A = B = \frac{1}{2}$). This is an instance of *initial conditions*, which we shall discuss shortly.

1.3.2 Elementary solutions of ordinary differential equations

Separable equations

A first-order differential equation is said to be **separable** if it can be written in the form $\frac{dy}{dx} = g(x)h(y)$, and the general solution of the equation may be obtained by evaluating the integrals in the equation

$$\int \frac{1}{h(y)} dy = \int g(x) dx.$$

Example 1.20

To obtain the solution of the equation $y' = xy$, we rewrite it in the form $y'/y = x$, then integrate to obtain $\int (y'/y) dx = \int dy/y = \int x dx$. Evaluating the integrals gives $\ln y = \frac{1}{2}x^2 + c$, where c is an arbitrary constant. In fact, the solution is made simpler by choosing the arbitrary constant to be $c = \ln A$, where $A > 0$ is also arbitrary, for then $\ln y = \frac{1}{2}x^2 + \ln A$, and the solution is $y = A \exp(\frac{1}{2}x^2)$. (This is a solution of the original differential equation for any choice of the constant A , i.e. it is the general solution.) ■

Example 1.21

The general solution of the differential equation $\frac{dy}{dx} = \frac{e^x}{1+y^2}$ may be found by evaluating $\int (1+y^2) dy = \int e^x dx$ to give $y + \frac{1}{3}y^3 = e^x + c$. This is an implicit equation for y . ■

The solution of a differential equation that satisfies certain additional constraints is known as a **particular solution**.

Example 1.22

To find a particular solution of the differential equation $y \frac{dy}{dx} + x = 0$ for which $y(0) = 3$, we first recognise that the equation is separable, then find the general solution from the equation $\int y dy = - \int x dx$. Thus the general solution is $x^2 + y^2 = c$, and it only remains to find the value of the constant c . Putting $y = 3$ when $x = 0$, we have $c = 0^2 + 3^2 = 9$, so the particular solution is $x^2 + y^2 = 9$.

Alternatively, we may notice that the solution may be written in terms of definite integrals so that the initial conditions are automatically satisfied.

We have $\int_3^y y \, dy = -\int_0^x x \, dx$, and evaluating the integrals gives the required result directly. This is a useful trick that is worth remembering. ■

Example 1.23

A boating pond is 10 m wide and 20 m long. A boy stands in one corner; his toy boat, attached to a string 10 m long, is at an adjacent corner, as shown in Figure 1.8. He walks along the long side of the pond and tows the boat, keeping the string taut. Find the equation of the path followed by the boat.

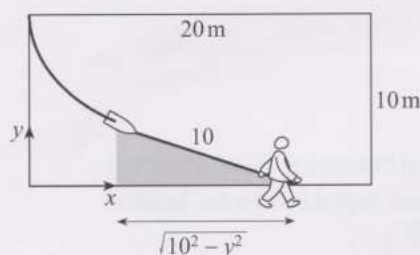


Figure 1.8 The path of a toy boat

Solution

We use axes as shown in the figure. When the boat is at the point (x, y) , the boy has moved a distance $x + \sqrt{100 - y^2}$, and the slope of the hypotenuse of the shaded triangle in Figure 1.8 is $-y/\sqrt{10^2 - y^2}$. Thus the gradient of the path followed by the boat is

$$\frac{dy}{dx} = -\frac{y}{\sqrt{100 - y^2}},$$

so

$$-\int dx = \int \frac{\sqrt{100 - y^2}}{y} dy.$$

The integral on the right-hand side can be evaluated using the substitution $u^2 = 100 - y^2$ for $u \geq 0$, which gives

$$\int \frac{\sqrt{100 - y^2}}{y} dy = \int \frac{u}{y} \left(-\frac{y}{u} du \right) = \int \frac{u^2}{u^2 - 100} du.$$

Now notice that $y = 10$, and therefore $u = 0$, when $x = 0$. So, using partial fractions and the trick mentioned in the previous example, we have

$$-\int_0^x dx = \int_0^u \frac{u^2}{u^2 - 100} du = \int_0^u \left(1 - 5 \left(\frac{1}{10 - u} + \frac{1}{10 + u} \right) \right) du.$$

Thus

$$-x = [u + 5 \ln(10 - u) - 5 \ln(10 + u)]_0^u = u + 5 \ln \left(\frac{10 - u}{10 + u} \right),$$

which gives

$$x = 5 \ln \left(\frac{10 + u}{10 - u} \right) - u = 5 \ln \left(\frac{10 + \sqrt{100 - y^2}}{10 - \sqrt{100 - y^2}} \right) - \sqrt{100 - y^2}.$$

In this example we cannot rearrange the equation to give $y = y(x)$, but we could use a computer program to plot the graph of $x = x(y)$ as shown in Figure 1.9.

Note that in an integral such as $\int_3^y f(y) \, dy$, we use the same variable for the integrand and the upper limit. This expression should be interpreted as

$$\left(\int f(x) \, dx \right) \Big|_{x=3}^{x=y}.$$

The 'shorthand' is frequently useful because it saves on substitution of variables.

The integration after the substitution uses the method of partial fractions as discussed in Subsection 1.2.5.

*improper
an long
division*

$$= 1 + \frac{100}{u^2 - 100}$$

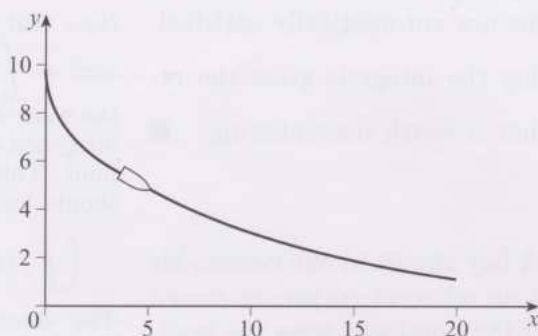


Figure 1.9 The graph of $x = x(y)$ (with the independent variable y on the vertical axis) ■

Homogeneous equations of first order

The word ‘homogeneous’ is commonly used in mathematics in several different senses. Here we define a first-order differential equation to be **homogeneous** if it is an equation of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

In order to solve such an equation, we write $y = xv$ and endeavour to find a differential equation in v and x that is separable. Differentiating with respect to x , we obtain $\frac{dy}{dx} = x\frac{dv}{dx} + v$, and eliminating y from the original equation gives $x\frac{dv}{dx} + v = f(v)$, which can be rearranged into the form $\frac{dv}{dx} = \frac{f(v) - v}{x}$, which is separable.

Example 1.24

In order to solve the equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy},$$

we first notice that it is a homogeneous first-order differential equation, since it can be written as

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} = f\left(\frac{y}{x}\right).$$

So we make the substitution $y = xv$, giving $y' = xv' + v$ as above. Eliminating y gives

$$x\frac{dv}{dx} + v = \frac{x}{y} + \frac{y}{x} = \frac{1}{v} + v,$$

and the differential equation becomes

$$x\frac{dv}{dx} = \frac{1}{v}, \quad \text{so} \quad \int v \, dv = \int \frac{dx}{x}.$$

Hence $\frac{1}{2}v^2 = \ln x + A$ for $x > 0$, and finally replacing v by y/x gives

$$\left(\frac{y}{x}\right)^2 = 2 \ln x + C,$$

where $C = 2A$ is an arbitrary constant. ■

Exercise 1.29

In each case, show that the equation is homogeneous, and solve it.

$$(a) \frac{dy}{dx} = \frac{x+y}{x-y} \quad (b) \quad 3xy^2 \frac{dy}{dx} = x^3 + y^3$$

Exercise 1.30

Show that the differential equation

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}$$

is homogeneous, and use the substitution $y = xv$ to solve it. [Hint: Try the further substitution $u = \sqrt{1 + v^2}$.]

Linear equations of first order

The general **linear equation of first order** takes the form

$$a_1(x) \frac{dy}{dx} + a_0(x)y = f(x).$$

For example, the equation

$$x^3 \frac{dy}{dx} + 3x^2 y = x^5$$

is of this form, and in this case it is easy to find the solution if we notice that

$$x^3 \frac{dy}{dx} + 3x^2 y = \frac{d}{dx}(x^3 y).$$

Then the differential equation becomes

$$\frac{d}{dx}(x^3 y) = x^5,$$

and we have only to integrate this expression to obtain the solution

$$x^3 y = \frac{1}{6} x^6 + c.$$

However, it is not immediately clear how we would proceed if the same equation were to arise in a slightly different form, for example, $\frac{dy}{dx} + 2xy = 6x$. Fortunately, there is a simple method that will transform such an equation into a form that can be integrated.

The integrating factor method

Any linear differential equation of first order of the form

$$\frac{dy}{dx} + g(x)y = h(x) \tag{1.10}$$

may be solved by multiplying both sides of the equation by the **integrating factor**

$$P(x) = \exp\left(\int g(x) dx\right).$$

The differential equation then becomes

$$\frac{d}{dx}(yP(x)) = h(x)P(x),$$

which may be integrated to give the general solution

$$y = \frac{1}{P(x)} \int h(x)P(x) dx.$$

Example 1.25

Find the general solution of the differential equation $\frac{dy}{dx} + 2xy = 6x$.

Solution

This equation is of the same form as equation (1.10) (i.e. first-order linear), with $g(x) = 2x$ and $h(x) = 6x$. The integrating factor is

$$P(x) = \exp\left(\int 2x \, dx\right) = \exp(x^2).$$

So the general solution is

$$\begin{aligned} y &= \frac{1}{\exp(x^2)} \int 6x \exp(x^2) \, dx = \frac{3}{\exp(x^2)} (\exp(x^2) + A) \\ &= 3 + 3A \exp(-x^2). \quad \blacksquare \end{aligned}$$

Example 1.26

Find the particular solution of the differential equation

$$R \frac{dq}{dt} + \frac{q}{C} = E(t),$$

where $R = 10$, $C = 10^{-3}$ and $E(t) = 20 \sin(100\pi t)$, that satisfies $q(0) = 0$.

Solution

The differential equation becomes

$$10 \frac{dq}{dt} + 10^3 q = 20 \sin(100\pi t),$$

which simplifies to

$$\frac{dq}{dt} + 100q = 2 \sin(100\pi t).$$

This equation is first-order linear (see equation (1.10)), with $g = 100$ and $h = 2 \sin(100\pi t)$. So the integrating factor is $P(t) = \exp\left(\int 100 \, dt\right) = e^{100t}$, and multiplying the differential equation by the integrating factor gives

$$\frac{d}{dt}(e^{100t} q) = 2e^{100t} \sin(100\pi t).$$

Integrating, we obtain

$$e^{100t} q = \frac{e^{100t}}{50(1 + \pi^2)} (\sin(100\pi t) - \pi \cos(100\pi t)) + C.$$

Using the condition $q = 0$ when $t = 0$ gives $C = \pi/50(1 + \pi^2)$, and finally

$$q = \frac{1}{50(1 + \pi^2)} (\sin(100\pi t) - \pi \cos(100\pi t) + \pi e^{-100t}). \quad \blacksquare$$

Differential equations of this kind arise in the theory of electrical circuits.

We discussed integrals of the form $\int e^{at} \sin(bt) \, dt$ on page 26.

Exercise 1.31

- Find the particular solution of the differential equation $\frac{dy}{dx} + y \cot x = 3e^{\cos x}$ that satisfies $y = -1$ when $x = \frac{\pi}{2}$.
- Find the general solution of the differential equation $(x-1)(x-2) \frac{dy}{dx} + y = x^2$.
- Show that the transformation $v = y^{1-n}$, where $n \neq 1$ is a constant, reduces the differential equation $\frac{dy}{dx} + yF(x) = y^n G(x)$ to linear form.

Linear equations with constant coefficients

Linear differential equations with constant coefficients are particularly easy to solve. We shall concentrate on equations of order two, but higher-order equations can be solved in a similar fashion. (For order greater than four, the auxiliary equation cannot generally be solved algebraically.)

We consider differential equations of the form

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x), \quad \text{where } a, b \text{ and } c \text{ are constants,}$$

and we assume that you have seen such equations in a previous course; these brief notes are intended primarily as revision.

Procedure

The above differential equation is said to be homogeneous when $f(x) = 0$. In order to solve the homogeneous equation

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

where a, b and c are real constants and $a \neq 0$, we use the following procedure.

- Write down the auxiliary equation $am^2 + bm + c = 0$, and find its roots m_1 and m_2 .
- If the auxiliary equation has two distinct roots, then the general solution is $y = Ae^{m_1 x} + Be^{m_2 x}$.
- If the auxiliary equation has two equal roots, i.e. $m_1 = m_2$, then the general solution is $y = (A + Bx)e^{m_1 x}$.
- If the auxiliary equation has a pair of complex conjugate roots, given by $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution is $y = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$.

(In each case, A and B are arbitrary constants.)

In order to find the **general solution** of the second-order inhomogeneous linear equation with constant coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x), \quad (1.11)$$

we first use the procedure above to find the general solution $y_c(x)$ of the associated homogeneous equation $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$. Then we find a particular solution $y_p(x)$ of equation (1.11). Here $y_c(x)$ is known as the **complementary function**, and $y_p(x)$ as a **particular integral**.

The general solution of equation (1.11) is then

$$y(x) = y_c(x) + y_p(x).$$

For an n th-order differential equation, y will be a solution containing n arbitrary constants, so it will indeed be the general solution.

There are various techniques for finding particular integrals y_p , and we refer you to any of the standard texts on differential equations for the details. Here we illustrate one of these techniques, known as the **method of undetermined coefficients**. In order to apply this method, we need to be able to guess the form of a solution and then determine the constants in our proposed form by equating coefficients. The following example illustrates the method.

Note that the term 'homogeneous' as used here differs from its application to first-order equations on page 34.

If the coefficients a, b and c are real, and the auxiliary equation has complex roots, then these roots will always occur in complex conjugate pairs.

Example 1.27

To find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 10,$$

we consider first the auxiliary equation $m^2 - 5m + 6 = 0$, which has roots $m_1 = 2$ and $m_2 = 3$. Hence the general solution of the homogeneous equation (i.e. the complementary function) is $y_c = Ae^{2x} + Be^{3x}$. In order to find a particular integral, we guess that a constant function $y_p(x) = k$ may be suitable, and substituting this into the original differential equation gives $6k = 10$, so $k = \frac{5}{3}$. It follows that the general solution of the differential equation is

$$y = y_c + y_p = Ae^{2x} + Be^{3x} + \frac{5}{3}.$$

This is the *general* solution of the second-order differential equation because it satisfies the equation (this can be checked by substitution), and it contains two arbitrary constants. ■

If the number 10 in the previous example were to be replaced by $10 + 3x + x^2$, then we might guess that $a + bx + cx^2$ is a suitable form for a particular integral.

The method may also be extended to differential equations of higher order.

Example 1.28

Find the general solution of the equation $\frac{d^4y}{dx^4} - y = 0$.

Solution

This equation is homogeneous, so it is not necessary to find a particular integral. We consider the auxiliary equation $m^4 - 1 = 0$: factorising the left-hand side gives

$$m^4 - 1 = (m^2 - 1)(m^2 + 1) = (m - 1)(m + 1)(m - i)(m + i),$$

so the general solution of the differential equation is

$$y = Ae^x + Be^{-x} + C \cos x + D \sin x.$$

This is the *general* solution of the fourth-order differential equation, because it satisfies the equation and contains four arbitrary constants. ■

Exercise 1.32

(a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$$

(b) Given that $y_1 = -xe^x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x,$$

find the general solution.

In each of Exercises 1.33–1.35, a particular integral may be found by using the method of undetermined coefficients. Try to guess the form of a solution, then equate coefficients to find a particular integral.

Exercise I.33

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 3 - 2x.$$

Exercise I.34

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = 2x.$$

Exercise I.35

Find the general solution of the differential equation

$$2\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = x^2 + 2x - 1.$$

Exercise I.36

Find the general solution of the differential equation

$$\frac{d^4y}{dx^4} + y = 0.$$

Initial conditions and boundary conditions

We have previously discussed cases where a particular solution of a differential equation is defined by the value of the function at a specific point, but sometimes we require more than one additional piece of information. As an illustration, consider the equation defining simple harmonic motion

$$\frac{d^2y}{dt^2} = -\omega^2 y, \quad (1.12)$$

where ω is a specified real positive constant, y denotes the displacement of an object, and the independent variable t represents time. The general solution of this equation is $y = A \cos \omega t + B \sin \omega t$, and we require two further conditions if we are to determine the values of the constants A and B . Often such conditions arise in one of two forms: **initial conditions**, in which the value of the function and its derivatives are specified at some chosen value of t , or **boundary conditions**, in which the function is defined at two different values of t . Boundary conditions will be dealt with later in the course.

Suppose that an object performs simple harmonic motion and satisfies equation (1.12), and that we are given the position y_0 and velocity v_0 of the object at time $t = 0$; in other words, we are given *initial conditions* for the motion of the object. We can easily determine that $A = y_0$ and $B = v_0/\omega$, and this provides us with the particular solution

$$y(t) = y_0 \cos \omega t + (v_0/\omega) \sin \omega t.$$

Here initial conditions are sufficient to determine a unique solution for any choice of y_0 and v_0 . This is generally the case; provided that we are given an appropriate number of initial conditions, the solution is always uniquely determined. An n th-order equation requires n initial conditions.

End-of-section Exercises

Exercise 1.37

Solve the equation $\frac{dy}{dx} = e^{2x+y}$, given that $y(0) = 0$.

Exercise 1.38

Solve the equation $\frac{dy}{dx} = -2x \tan y$, given that $y(0) = \frac{\pi}{2}$.

Exercise 1.39

Solve the equation $\frac{dy}{dx} = \frac{2xy}{3x^2 - 2y^2}$, given that $y(1) = 1$. [Hint: Show that the equation is homogeneous, and use the method of partial fractions.]

Exercise 1.40

Solve the equation $x \frac{dy}{dx} = y(1 + \ln y - \ln x)$. [Hint: Show that the equation is homogeneous, and solve the resulting integral by a suitable change of variable.]

Exercise 1.41

Solve the equation $x \left(\frac{dy}{dx} \right)^2 - 2y \frac{dy}{dx} - x = 0$. [Hint: First solve a quadratic equation, then remember that $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$.]

Exercise 1.42

Find the general solution of the equation $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 12x^2 + 10x - 11$.

Exercise 1.43

Harder exercise

- (a) Find the family of curves for which the length of the part of the tangent between the point of contact (x, y) , in the first quadrant, and the y -axis is equal to the y -coordinate of the intercept of this tangent with the y -axis. (In other words, find curves where the lengths OA and AP shown in Figure 1.10 are equal.)

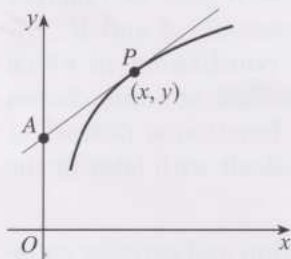


Figure 1.10

- (b) Find the equation of the orthogonal family of curves (i.e. the curves such that the product of the gradients at the point of intersection with each member of the previous family is equal to -1).

Exercise 1.44

Harder exercise

A chemical compound C is being formed by the reaction of two substances, A and B. During the reaction, substance A combines with substance B in the ratio 2 : 1 by mass to form substance C. The rate of formation of C is proportional to the product of the masses of the quantities of A and B that remain uncombined. Initially, there are 200 grams of A, 100 grams of B, and 0 grams of C in the mixture. Given that 100 grams of substance C are formed after 30 seconds, find the amount $z(t)$ of substance C formed after time t .

1.4 Outcomes

After studying this chapter you should be able to:

- understand the terms *continuous*, *differentiable* and *order* when applied to functions, and apply the O notation;
- understand what is meant by the convergence of an infinite series;
- differentiate expressions involving elementary functions;
- evaluate the Taylor series for a function, and understand what is meant by its radius of convergence;
- integrate elementary functions;
- understand what is meant by the convergence of an integral over an infinite range;
- express rational functions as partial fractions, and apply this technique to the integration of rational functions and the expansion of rational functions in series;
- manipulate expressions involving complex numbers;
- transform between the Cartesian, polar and exponential forms of a complex number;
- calculate powers and roots of complex numbers using de Moivre's theorem;
- understand the terms *ordinary*, *partial*, *linear*, *order*, *general solution* and *particular solution* when applied to differential equations;
- solve first-order ordinary differential equations which are separable, homogeneous or linear;
- solve higher-order ordinary differential equations which are linear with constant coefficients;
- apply initial conditions to general solutions, to fix the arbitrary constants.

Solutions to Exercises in Chapter 1

Solution 1.1

We have

$$R_n(a) = \sum_{k=n}^{\infty} a^k = a^n + a^{n+1} + a^{n+2} + \cdots = a^n(1 + a + a^2 + \cdots).$$

For $-1 < a < 1$, using equation (1.2) gives

$$R_n(a) = a^n \lim_{k \rightarrow \infty} S_k(a) = \frac{a^n}{1-a}.$$

Solution 1.2

From Equation (1.1) we have

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Replacing x by $-x^2$ gives

$$\sum_{k=0}^N (-1)^k x^{2k} = \frac{1 + (-1)^{N+1} x^{2N+2}}{1 + x^2},$$

which converges to the limit $\frac{1}{1+x^2}$ as $N \rightarrow \infty$ if $|x| < 1$, but diverges if $|x| > 1$.

Thus the infinite series $\sum_{k=0}^{\infty} (-1)^k x^{2k}$ converges if $|x| < 1$, but diverges if $|x| > 1$, so the largest interval $|x| < \rho$ corresponds to choosing $\rho = 1$.

This exercise is one of the very rare occasions when we are actually able to determine the function f in a simple form from the coefficients $\{a_k\}$, for in this case

$$f(x) = \lim_{N \rightarrow \infty} \left[\frac{1 + (-1)^{N+1} x^{2N+2}}{1 + x^2} \right] = \frac{1}{1 + x^2}.$$

Solution 1.3

From the Taylor series we have $\sin x = x + O(x^3)$ and $\ln(1+x) = x + O(x^2)$. It follows that

$$\sin x \ln(1+x) = [x + O(x^3)][x + O(x^2)] = x^2 + O(x^3).$$

Solution 1.4

We have

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 \\ &= \left(\frac{e^{2x} + 2 + e^{-2x}}{4} \right) - \left(\frac{e^{2x} - 2 + e^{-2x}}{4} \right) = 1. \end{aligned}$$

Solution 1.5

(a) $\ln x = \int_1^x \frac{1}{t} dt$ and $\frac{1}{t} \leq 1$ for $t \geq 1$. From the stated result we therefore have

$$\ln x = \int_1^x \frac{1}{t} dt \leq x - 1 < x.$$

(b) The formula for integration by parts may be written in the form

$$\int u(x) v'(x) dx = u(x) v(x) - \int u'(x) v(x) dx.$$

- (i) Choosing $u(x) = \ln x$ and $v(x) = \ln x$, we have

$$\int \frac{\ln x}{x} dx = (\ln x)^2 - \int \frac{\ln x}{x} dx,$$

which can be rearranged to give

$$\int \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2.$$

Alternatively, make the substitution $s = \ln x$; then

$$\int \frac{\ln x}{x} dx = \int s ds = \frac{1}{2}s^2 = \frac{1}{2}(\ln x)^2.$$

So

$$\int_1^T \frac{\ln x}{x} dx = \frac{1}{2}(\ln T)^2.$$

- (ii) Choosing $u(x) = \ln x$ and $v(x) = -x^{-1}$, we have

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x}.$$

So

$$\int_1^T \frac{\ln x}{x^2} dx = \left(-\frac{\ln T}{T} - \frac{1}{T} \right) - (-1) = 1 - \frac{1}{T} - \frac{\ln T}{T}.$$

- (c) (i) Using the result of part (b)(i), we have

$$\int_1^T \frac{\ln x}{x} dx = \frac{1}{2}(\ln T)^2,$$

which increases without bound as T increases, so the integral is divergent.

- (ii) From part (b)(ii),

$$\int_1^T \frac{\ln x}{x^2} dx = 1 - \frac{1}{T} - \frac{\ln T}{T} \leq 1,$$

so the integral is bounded. Since also $\frac{\ln x}{x^2} \geq 0$, the integral is convergent.

Solution 1.6

For $1 \leq x \leq T$, we have $0 < e^{-x^2} \leq xe^{-x^2}$, so using the stated result gives

$$\int_1^T e^{-x^2} dx \leq \int_1^T xe^{-x^2} dx = \left[-\frac{1}{2}e^{-x^2} \right]_1^T = -\frac{1}{2}e^{-T^2} + \frac{1}{2e} \leq \frac{1}{2e}.$$

Since this integral is bounded, and $e^{-x^2} > 0$, we see that the integral $\int_1^\infty e^{-x^2} dx$ is convergent. Using the hint, the convergence of $\int_0^\infty e^{-x^2} dx$ follows immediately.

Solution 1.7

- (a) We write

$$\frac{11x+5}{(2x-1)(3x+2)} = \frac{A}{2x-1} + \frac{B}{3x+2}.$$

Then putting $x = 0$ gives $-5/2 = -A + B/2$, and putting $x = 1$ gives $16/5 = A + B/5$. Solving this pair of equations for A and B , we obtain $A = 3$ and $B = 1$, so

$$\frac{11x+5}{(2x-1)(3x+2)} = \frac{3}{2x-1} + \frac{1}{3x+2}.$$

(b) We write

$$\frac{2(x+1)(2x+7)}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3}. \quad (1.13)$$

By substituting suitable values for x , or multiplying both sides by the denominator $(x-1)(x+2)(x+3)$ and equating coefficients of powers of x , we find

$$A = 3, \quad B = 2, \quad C = -1.$$

There is a third method which is often useful. Suppose that we multiply both sides of expression (1.13) by $x-1$, to obtain

$$\frac{2(x+1)(2x+7)}{(x+2)(x+3)} = A + \frac{B(x-1)}{x+2} + \frac{C(x-1)}{x+3},$$

and then put $x = 1$. We obtain $A = \frac{2 \times 2 \times 9}{3 \times 4} = 3$ immediately. Similarly, multiplying (1.13) by $x+2$ and putting $x = -2$, and multiplying by $x+3$ and putting $x = -3$, gives B and C at once.

(c) You may have noticed that in the given expression, the numerator is not of lower degree than the denominator, so our first step must be to overcome this difficulty. This can be achieved by writing

$$\begin{aligned} \frac{x^3 + 2x}{x^2 - 1} &= \frac{(x(x^2 - 1) + x) + 2x}{x^2 - 1} \\ &= \frac{x(x^2 - 1) + 3x}{(x^2 - 1)} \\ &= x + \frac{3x}{(x-1)(x+1)}. \end{aligned}$$

Proceeding as before, we find

$$\frac{x^3 + 2x}{x^2 - 1} = x + \frac{3}{2(x-1)} + \frac{3}{2(x+1)}.$$

Solution 1.8

$$(a) \quad \frac{x}{x^2 - 5x + 6} = \frac{3}{x-3} - \frac{2}{x-2}$$

(b) If

$$\frac{x^2}{(x-1)^2} = 1 + \frac{P(x)}{Q(x)},$$

then

$$\frac{P(x)}{Q(x)} = \frac{x^2}{(x-1)^2} - 1 = \frac{2x-1}{(x-1)^2} = \frac{2(x-1)+1}{(x-1)^2} = \frac{2}{x-1} + \frac{1}{(x-1)^2},$$

so

$$\frac{x^2}{(x-1)^2} = 1 + \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

$$(c) \quad \frac{(x-1)^2}{(x-3)^2(x-2)} = \frac{4}{(x-3)^2} + \frac{1}{x-2}$$

$$(d) \quad \frac{x-2}{(2+x^2)(1+x)^2} = \frac{x+2}{3(2+x^2)} - \frac{1}{3(1+x)} - \frac{1}{(1+x)^2}$$

Solution 1.9

$$z = \frac{a+bi}{c+di} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di} = \frac{1}{c^2+d^2} ((ac+bd) + (bc-ad)i)$$

Solution I.10

$z = -16 + 12i$ (and notice that it would be equally acceptable to write $z = 12i - 16$).

$|z| = \sqrt{(-16)^2 + 12^2} = 4\sqrt{4^2 + 3^2} = 20$ and $\text{Im}(z) = 12$ (not $12i$).

$z^* = -16 - 12i$ and $zz^* = |z|^2 = 400$.

Solution I.11

From de Moivre's theorem we have $(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$. So, from the binomial theorem and using the fact that $\cos^2 \theta + \sin^2 \theta = 1$, we have

$$\begin{aligned}\cos 5\theta &= \text{Re}((\cos \theta + i \sin \theta)^5) \\ &= \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta \\ &= \cos \theta [\cos^4 \theta - 10 \cos^2 \theta (1 - \cos^2 \theta) + 5(1 - \cos^2 \theta)^2] \\ &= \cos \theta (16 \cos^4 \theta - 20 \cos^2 \theta + 5).\end{aligned}$$

Solution I.12

- (a) First we calculate $|z| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$, then if we plot the point $z = 2 - 2i$ on an Argand diagram, we can easily see that $\text{Arg}(z) = -\frac{\pi}{4}$. Thus, in polar form, we have

$$z = 2\sqrt{2} \left[\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right].$$

Any value of the argument of the form $-\frac{\pi}{4} + 2k\pi$, where k is an integer, is equally acceptable.

- (b) The general polar form is $z = 2\sqrt{2} (\cos(-\frac{\pi}{4} + 2k\pi) + i \sin(-\frac{\pi}{4} + 2k\pi))$, for k an integer. Hence, using de Moivre's theorem, we obtain

$$z^{1/3} = 8^{1/6} \left(\cos\left(\frac{1}{3}\left(-\frac{\pi}{4} + 2k\pi\right)\right) + i \sin\left(\frac{1}{3}\left(-\frac{\pi}{4} + 2k\pi\right)\right) \right).$$

For $k = 0, 1, 2$, this gives three distinct cube roots:

$$\begin{aligned}z_0 &= \sqrt{2} \left(\cos\left(-\frac{\pi}{12}\right) + i \sin\left(-\frac{\pi}{12}\right) \right), \\ z_1 &= \sqrt{2} \left(\cos\frac{7\pi}{12} + i \sin\frac{7\pi}{12} \right), \\ z_2 &= \sqrt{2} \left(\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4} \right).\end{aligned}$$

The remaining values of k simply repeat these roots.

Solution I.13

For any real number n ,

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta).$$

Solution I.14

We have

$$|a + b|^2 = (a + b)^2 = a^2 + b^2 + 2ab \leq |a|^2 + |b|^2 + 2|a||b| = (|a| + |b|)^2,$$

and the required result follows.

Solution I.15

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh x, \\ \frac{d}{dx}(\cosh x) &= \frac{d}{dx} \left(\frac{e^x + e^{-x}}{2} \right) = \frac{e^x - e^{-x}}{2} = \sinh x.\end{aligned}$$

If $\phi = A \sinh \omega x + B \cosh \omega x$, then $\frac{d\phi}{dx} = A\omega \cosh \omega x + B\omega \sinh \omega x$, so

$$\frac{d^2\phi}{dx^2} = A\omega^2 \sinh \omega x + B\omega^2 \cosh \omega x = \omega^2 \phi.$$

Solution 1.16

We can write $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, therefore

$$\begin{aligned}\sum_{n=1}^N \frac{1}{n(n+1)} &= \sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1} \right) \\ &= 1 - \frac{1}{N+1}.\end{aligned}$$

This follows because each of the negative terms, except the last, cancels with the first term in the next bracket. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{n(n+1)} = \lim_{N \rightarrow \infty} \left(1 - \frac{1}{N+1} \right) = 1.$$

Solution 1.17

We know that

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!},$$

so

$$\sinh(2x) = \sum_{n=0}^{\infty} \frac{(2x)^{2n+1}}{(2n+1)!}.$$

Solution 1.18

From the Taylor series we have

$$\sin x = x - \frac{x^3}{3!} + O(x^5) \quad \text{and} \quad \frac{1}{1-x^2} = 1 + x^2 + O(x^4).$$

Hence

$$\begin{aligned}\frac{\sin x}{1-x^2} &= \left[x - \frac{1}{6}x^3 + O(x^5) \right] \left[1 + x^2 + O(x^4) \right] \\ &= x - \frac{1}{6}x^3 + x^3 + O(x^5) \\ &= x + \frac{5}{6}x^3 + O(x^5).\end{aligned}$$

Solution 1.19

We have

$$f(-x) = \frac{(-x)}{1+(-x)^2} = -\frac{x}{1+x^2} = -f(x),$$

so the function is odd. Thus

$$\begin{aligned}\int_{-1}^1 f(x) dx &= \int_0^1 f(x) dx + \int_{-1}^0 f(x) dx \\ &= \int_0^1 f(x) dx - \int_{-1}^0 f(-x) dx \\ &= \int_0^1 f(x) dx - \int_0^1 f(x) dx \\ &= 0.\end{aligned}$$

Here we have substituted x for $-x$ in the second integral.

For any odd function $f(x)$, and any real number a , it is obvious from the graph of f that $\int_{-a}^a f(x) dx = 0$.

Solution 1.20

If $w = \sinh^{-1} x$, then $x = \sinh w$. Differentiating with respect to x , we obtain

$$1 = \cosh w \frac{dw}{dx}.$$

It follows that

$$\frac{dw}{dx} = \frac{1}{\cosh w} = \frac{1}{\sqrt{1 + \sinh^2 w}} = \frac{1}{\sqrt{1 + x^2}}.$$

The graph of $y = \sinh^{-1} x$ can be obtained easily by reflecting the graph of $y = \sinh x$ in the line $y = x$.

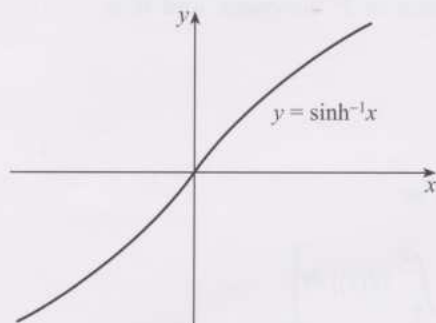


Figure 1.11 The graph of $\sinh^{-1} x$

We have $\sqrt{1+x^2} > \sqrt{x^2} \geq -x$ for all $x \in \mathbb{R}$, so $x + \sqrt{1+x^2} > 0$.

Now

$$\frac{d}{dx} \left(\ln \left(x + \sqrt{1+x^2} \right) \right) = \frac{1}{\sqrt{1+x^2}}.$$

So we see that $\frac{dw}{dx} = \frac{d}{dx} \left(\ln \left(x + \sqrt{1+x^2} \right) \right)$, therefore w and $\ln \left(x + \sqrt{1+x^2} \right)$ must differ by a constant, giving

$$\sinh^{-1} x = \ln \left(x + \sqrt{1+x^2} \right) + c.$$

However, $\sinh^{-1} x = \ln \left(x + \sqrt{1+x^2} \right) = 0$ when $x = 0$, so $c = 0$, and this establishes the required result.

Solution 1.21

We can write

$$\frac{4x^2 - 9x + 4}{x(x-1)(x-2)} = \frac{2}{x} + \frac{1}{x-1} + \frac{1}{x-2},$$

so for $x > 2$ we have

$$\begin{aligned} \int \frac{4x^2 - 9x + 4}{x(x-1)(x-2)} dx &= \int \left(\frac{2}{x} + \frac{1}{x-1} + \frac{1}{x-2} \right) dx \\ &= 2 \ln x + \ln(x-1) + \ln(x-2) + c \\ &= \ln(x^2(x-1)(x-2)) + c. \end{aligned}$$

It follows that

$$\int_3^4 \frac{4x^2 - 9x + 4}{x(x-1)(x-2)} dx = \left[\ln(x^2(x-1)(x-2)) \right]_3^4 = \ln(16/3).$$

Solution 1.22

For $a \neq 1$ we have

$$\int_1^T \frac{1}{t^a} dt = \frac{1}{1-a} [t^{1-a}]_1^T = \frac{1}{1-a} (T^{1-a} - 1),$$

so $\lim_{T \rightarrow \infty} \int_1^T \frac{1}{t^a} dt = \frac{1}{a-1}$ if $a > 1$, while for $a < 1$ the limit does not exist.

We also showed in a previous example that $\int_1^\infty \frac{1}{t} dt$ is divergent.

It follows that $\int_1^\infty \frac{1}{t^a} dt$ is convergent for $a > 1$ and divergent for $a \leq 1$.

Solution 1.23

First we notice that $0 \leq f(t) + |f(t)| \leq 2|f(t)|$, so

$$\int_a^T (f(t) + |f(t)|) dt \leq 2 \int_a^T |f(t)| dt \leq 2 \int_a^\infty |f(t)| dt.$$

Thus the integral $\int_a^T (f(t) + |f(t)|) dt$ is non-decreasing as T increases, and it is bounded above, so the limit

$$\lim_{T \rightarrow \infty} \int_a^T (f(t) + |f(t)|) dt$$

exists. But we are given that $\lim_{T \rightarrow \infty} \int_a^T |f(t)| dt$ exists, so

$$\lim_{T \rightarrow \infty} \int_a^T f(t) dt = \lim_{T \rightarrow \infty} \left[\int_a^T (f(t) + |f(t)|) dt - \int_a^T |f(t)| dt \right]$$

exists. In other words, $\int_a^\infty f(t) dt$ is convergent.

Since $\left| \frac{\sin t}{t^2} \right| \leq \frac{1}{t^2}$ (for $t \neq 0$) and $\int_1^\infty \frac{1}{t^2} dt$ is convergent, it follows that $\int_1^\infty \left| \frac{\sin t}{t^2} \right| dt$ is convergent, therefore $\int_1^\infty \frac{\sin t}{t^2} dt$ is also convergent.

Solution 1.24

- (a) $i - 1$ (b) $\frac{3}{2} + i$ (c) 1

Solution 1.25

- (a) $1 - i = \sqrt{2}e^{-i\pi/4}$, so $\frac{1}{1-i} = \frac{1}{\sqrt{2}}e^{i\pi/4}$.
 (b) $(1 + i\sqrt{3})^2 = (2e^{i\pi/3})^2 = 4e^{2i\pi/3}$
 (c) $(1 + i\sqrt{3})^{15} = (2e^{i\pi/3})^{15} = 2^{15}e^{15i\pi/3} = 2^{15}e^{5i\pi} = 2^{15}e^{i\pi}$

The polar forms of these complex numbers can be obtained by using Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$.

Solution 1.26

$$\begin{aligned} \cosh(i\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{(\cos \theta + i \sin \theta) + (\cos \theta - i \sin \theta)}{2} = \cos \theta, \\ \sinh(i\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{(\cos \theta + i \sin \theta) - (\cos \theta - i \sin \theta)}{2} = i \sin \theta. \end{aligned}$$

Solution 1.27

$$\begin{aligned} |z+w|^2 + |z-w|^2 &= (z+w)(z+w)^* + (z-w)(z-w)^* \\ &= (z+w)(z^*+w^*) + (z-w)(z^*-w^*) \\ &= 2zz^* + 2ww^* \\ &= 2|z|^2 + 2|w|^2 \end{aligned}$$

Solution 1.28

- (a) (i) If $y = e^{ax}$, then $\ln y = ax$, and differentiating with respect to x gives $a = y'/y$. Substituting y'/y for a , and using the fact that $y \neq 0$, we obtain the first-order equation

$$y \ln y = xy'.$$

- (ii) Differentiating gives $y' = \cos(x+a)$; using the identity $\cos^2 \theta + \sin^2 \theta = 1$, we have

$$(y')^2 + y^2 = 1.$$

- (b) Since $(x+b)y = x+a$, differentiating with respect to x gives $(x+b)y' + y = 1$. Differentiating again, we have $(x+b)y'' + 2y' = 0$. Eliminating $x+b$ from these two equations gives

$$2(y')^2 = (y-1)y''.$$

- (c) Integrating the equation $\frac{d^2y}{dx^2} = 1$ twice, we obtain first $\frac{dy}{dx} = x + A$, then $y = \frac{1}{2}x^2 + Ax + B$, where A and B are arbitrary constants.

- (d) Differentiating the equation $xy + \cos y = 3$ gives $x\frac{dy}{dx} + y - \sin y\frac{dy}{dx} = 0$, which can be rearranged to give $\frac{dy}{dx} = \frac{y}{\sin y - x}$.

Solution 1.29

- (a) The equation can be written as

$$\frac{dy}{dx} = \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}},$$

so it is homogeneous.

Putting $y = xv$ as before, and eliminating y , we obtain the equation

$$x\frac{dv}{dx} + v = \frac{1+v}{1-v},$$

which can be rearranged to give

$$\int \left(\frac{1}{1+v^2} - \frac{v}{1+v^2} \right) dv = \int \frac{dx}{x}.$$

Integrating gives

$$\tan^{-1} v - \frac{1}{2} \ln(1+v^2) = \ln x + A$$

for $x > 0$, and replacing v by y/x gives

$$2 \tan^{-1} \left(\frac{y}{x} \right) - \ln \left(\frac{x^2 + y^2}{x^2} \right) = 2 \ln x + B,$$

that is,

$$2 \tan^{-1}(y/x) - \ln(x^2 + y^2) = B.$$

- (b) The equation can be written as

$$\frac{dy}{dx} = \frac{1}{3} \left(\frac{x}{y} \right)^2 + \frac{1}{3} \left(\frac{y}{x} \right),$$

so it is homogeneous.

Putting $y = xv$, the equation becomes

$$x\frac{dv}{dx} + v = \frac{1}{3} \left(\frac{1}{v^2} + v \right),$$

which can be rearranged to give

$$x\frac{dv}{dx} = \frac{1-2v^3}{3v^2}.$$

Rearranging again and integrating, we have

$$\int \frac{3v^2}{1-2v^3} dv = \int \frac{dx}{x},$$

hence $\frac{1}{2} \ln(1-2v^3) = -\ln x + \frac{1}{2} \ln A$ (for $x > 0$, $2v^3 < 1$), which can be rewritten more simply as $1-2v^3 = A/x^2$, so the required solution is

$$x^3 - 2y^3 = Ax.$$

The choice of the arbitrary constant in the form $\frac{1}{2} \ln A$ simplifies the final answer.

Solution 1.30

The equation can be written as

$$\frac{dy}{dx} = -\frac{-1 \pm \sqrt{1 + (y/x)^2}}{(y/x)},$$

so it is homogeneous. Putting $y = xv$, we have

$$x \frac{dv}{dx} + v = \frac{-1 \pm \sqrt{1 + v^2}}{v},$$

which can be rearranged to give

$$x \frac{dv}{dx} = \frac{-v^2 - 1 \pm \sqrt{1 + v^2}}{v}$$

and therefore

$$\int \frac{dx}{x} = \int \frac{v dv}{-(1 + v^2) \pm \sqrt{1 + v^2}}.$$

The integral on the right can be evaluated by making the substitution $u = \sqrt{1 + v^2}$, and we then have

$$\int \frac{dx}{x} = \int \frac{du}{-u \pm 1}.$$

Integrating, we obtain

$$\ln x = -\ln(-u \pm 1) + \ln C,$$

so

$$x = \frac{C}{\pm 1 - u} = \frac{C}{\pm 1 - \sqrt{1 + v^2}} = \frac{Cx}{\pm x - \sqrt{x^2 + y^2}}.$$

Therefore $\pm x - \sqrt{x^2 + y^2} = C$, which can be rearranged to give

$$y^2 = C(C \mp 2x).$$

Solution 1.31

(a) The equation is first-order linear, and the integrating factor is

$$P(x) = \exp\left(\int \cot x dx\right) = \exp(\ln(\sin x)) = \sin x.$$

Multiplying the differential equation by the integrating factor gives

$$\sin x \frac{dy}{dx} + y \cos x = \frac{d}{dx}(y \sin x) = 3e^{\cos x} \sin x,$$

and integrating gives

$$y \sin x = \int 3e^{\cos x} \sin x dx = -3e^{\cos x} + A.$$

Since $y = -1$ when $x = \frac{\pi}{2}$, we have $(-1) \sin \frac{\pi}{2} = -3e^{\cos \frac{\pi}{2}} + A$, which gives $A = 2$, so the required solution is

$$y \sin x = -3e^{\cos x} + 2.$$

(b) In order to solve the differential equation

$$(x-1)(x-2) \frac{dy}{dx} + y = x^2,$$

we first rearrange it into the form

$$\frac{dy}{dx} + \frac{1}{(x-1)(x-2)} y = \frac{x^2}{(x-1)(x-2)}.$$

This is of the same form as equation (1.10) on page 35 (i.e. first-order linear), with

$$g(x) = \frac{1}{(x-1)(x-2)} = \frac{1}{x-2} - \frac{1}{x-1},$$

Notice that we must ensure that the coefficient of dy/dx is 1 before using the formula for the integrating factor. It is a common error to forget to divide the equation

$$a_1(x) \frac{dy}{dx} + a_0(x)y = f(x)$$

by $a_1(x)$.

so the integrating factor is

$$\begin{aligned} P(x) &= \exp \left(\int g(x) dx \right) \\ &= \exp \int \left(\frac{1}{x-2} - \frac{1}{x-1} \right) dx \\ &= \exp (\ln(x-2) - \ln(x-1)) = \frac{x-2}{x-1}. \end{aligned}$$

Now we obtain a further rearrangement of the differential equation by multiplying both sides by $P(x)$ to give

$$\left(\frac{x-2}{x-1} \right) \frac{dy}{dx} + \frac{1}{(x-1)^2} y = \frac{x^2}{(x-1)^2},$$

which can be rewritten in the form

$$\frac{d}{dx} \left(\frac{x-2}{x-1} y \right) = 1 + \frac{2}{x-1} + \frac{1}{(x-1)^2},$$

where we have expanded the right-hand side in terms of partial fractions. Integrating gives the general solution

$$\frac{x-2}{x-1} y = x + 2 \ln(x-1) - \frac{1}{x-1} + c,$$

which can be rearranged to give

$$y = \frac{x-1}{x-2} \left(x + 2 \ln(x-1) - \frac{1}{x-1} + c \right).$$

- (c) Differentiating $v = y^{1-n}$ with respect to x gives

$$(1-n) \frac{dy}{dx} = y^n \frac{dv}{dx},$$

and the differential equation becomes

$$\frac{y^n}{1-n} \frac{dv}{dx} + yF(x) = y^n G(x).$$

This can be rearranged into the form

$$\frac{dv}{dx} + (1-n)y^{1-n}F(x) = (1-n)G(x),$$

which then becomes

$$\frac{dv}{dx} + (1-n)vF(x) = (1-n)G(x),$$

which is of linear form (and can be solved by the previous method).

Solution 1.32

- (a) The auxiliary equation $m^2 + m - 6 = 0$ has roots $m_1 = 2$ and $m_2 = -3$, so the general solution is $y = Ae^{2x} + Be^{-3x}$.
- (b) The auxiliary equation $m^2 - 3m + 2 = 0$ has roots $m_1 = 1$ and $m_2 = 2$, so the general solution is $y = Ae^x + Be^{2x} - xe^x$.

Solution 1.33

The auxiliary equation $m^2 + 5m + 4 = 0$ has roots $m_1 = -1$ and $m_2 = -4$, so the complementary function is $y_c = Ae^{-x} + Be^{-4x}$.

The form of the right-hand side suggests that we should try a particular integral of the form $y_p = a + bx$. This gives

$$\frac{d^2 y_p}{dx^2} + 5 \frac{dy_p}{dx} + 4y_p = 5b + 4(a + bx) = (4a + 5b) + 4bx = 3 - 2x.$$

Equating coefficients of x and constant terms gives $a = \frac{11}{8}$ and $b = -\frac{1}{2}$. Thus the general solution is

$$y = y_c + y_p = Ae^{-x} + Be^{-4x} + \frac{11}{8} - \frac{1}{2}x.$$

Solution 1.34

The auxiliary equation $m^2 + 4 = 0$ has roots $m_1 = 2i$ and $m_2 = -2i$, so the general solution of the associated homogeneous equation is $y_c(x) = A \cos(2x) + B \sin(2x)$.

We now try a particular integral of the form $y_p = ax + b$, which gives $4(ax + b) = 2x$, from which we see that $a = \frac{1}{2}$ and $b = 0$. Thus the general solution is

$$y(x) = A \cos(2x) + B \sin(2x) + \frac{1}{2}x.$$

Solution 1.35

The auxiliary equation $2m^2 + 2m + 3 = 0$ has complex roots $m_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{5}i$ and $m_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{5}i$, so the general solution of the associated homogeneous equation is

$$y_c(x) = e^{-x/2} \left(A \cos\left(\frac{1}{2}\sqrt{5}x\right) + B \sin\left(\frac{1}{2}\sqrt{5}x\right) \right).$$

We now try a particular integral of the form $y_p = ax^2 + bx + c$, which gives

$$4a + 2(2ax + b) + 3(ax^2 + bx + c) = x^2 + 2x - 1.$$

Equating coefficients of powers of x gives $a = \frac{1}{3}$, $b = \frac{2}{9}$ and $c = -\frac{25}{27}$, so the general solution is

$$y(x) = e^{-x/2} \left(A \cos\left(\frac{1}{2}\sqrt{5}x\right) + B \sin\left(\frac{1}{2}\sqrt{5}x\right) \right) + \frac{1}{3}x^2 + \frac{2}{9}x - \frac{25}{27}.$$

Solution 1.36

We consider the auxiliary equation $m^4 + 1 = 0$, for which the solutions are the four possible values of $(-1)^{1/4}$. Writing -1 in polar form,

$$-1 = \cos(2k+1)\pi + i \sin(2k+1)\pi,$$

for any integer k , and using de Moivre's theorem, we have

$$\begin{aligned} (-1)^{1/4} &= (\cos(2k+1)\pi + i \sin(2k+1)\pi)^{1/4} \\ &= \cos\left(\frac{1}{4}(2k+1)\pi\right) + i \sin\left(\frac{1}{4}(2k+1)\pi\right). \end{aligned}$$

Choosing $k = 0, 1, 2, 3$, we obtain the four values

$$\begin{aligned} m_1 &= \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{1+i}{\sqrt{2}}, & m_2 &= \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} = \frac{i-1}{\sqrt{2}}, \\ m_3 &= \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} = -\frac{1+i}{\sqrt{2}}, & m_4 &= \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} = \frac{1-i}{\sqrt{2}}. \end{aligned}$$

(It is easy to verify that each of these is a solution of the equation $m^4 + 1 = 0$.)

The general solution of the differential equation is therefore

$$\begin{aligned} y &= \left(P \cos(x/\sqrt{2}) + Q \sin(x/\sqrt{2}) \right) e^{x/\sqrt{2}} \\ &\quad + \left(R \cos(x/\sqrt{2}) + S \sin(x/\sqrt{2}) \right) e^{-x/\sqrt{2}}. \end{aligned}$$

Solution 1.37

The equation is separable, and we have

$$\int e^{-y} dy = \int e^{2x} dx.$$

Using the initial condition $y(0) = 0$, we have

$$\int_0^y e^{-y} dy = \int_0^x e^{2x} dx,$$

so $[-e^{-y}]_0^y = \frac{1}{2} [e^{2x}]_0^x$, which gives $1 - e^{-y} = \frac{1}{2}(e^{2x} - 1)$, thus

$$y = \ln \left(\frac{2}{3 - e^{2x}} \right).$$

Solution 1.38

The equation is separable, and we have

$$\int \cot y \, dy = -2 \int x \, dx.$$

Using the initial condition $y(0) = \frac{\pi}{2}$, we have

$$\int_{\pi/2}^y \cot y \, dy = -2 \int_0^x x \, dx,$$

so $[\ln(\sin y)]_{\pi/2}^y = -[x^2]_0^x$, which gives $\ln(\sin y) = -x^2$, thus

$$y = \sin^{-1}(\exp(-x^2)).$$

Solution 1.39

Dividing the top and bottom of the right-hand side of the equation by xy , we obtain

$$\frac{dy}{dx} = \frac{2}{3x/y - 2y/x},$$

so the equation is homogeneous. Putting $y = xv$ gives $y' = xv' + v$, and the equation becomes $x \frac{dv}{dx} + v = \frac{2v}{3 - 2v^2}$. Rearranging this, we obtain

$$x \frac{dv}{dx} = \frac{v(2v^2 - 1)}{3 - 2v^2},$$

which is separable. Therefore, using the initial condition $v(1) = 1$, we have

$$\int_1^x \frac{dx}{x} = \int_1^v \frac{3 - 2v^2}{v(2v^2 - 1)} \, dv.$$

The integral on the right can be evaluated by writing the integrand in partial fractions as

$$\frac{3 - 2v^2}{v(2v^2 - 1)} = \frac{\sqrt{2}}{\sqrt{2}v - 1} + \frac{\sqrt{2}}{\sqrt{2}v + 1} - \frac{3}{v},$$

thus

$$\begin{aligned} \ln x &= \int_1^v \left(\frac{\sqrt{2}}{\sqrt{2}v - 1} + \frac{\sqrt{2}}{\sqrt{2}v + 1} - \frac{3}{v} \right) dv \\ &= [\ln(\sqrt{2}v - 1) + \ln(\sqrt{2}v + 1) - 3 \ln v]_1^v \\ &= \left[\ln \left(\frac{2v^2 - 1}{v^3} \right) \right]_1^v \\ &= \ln \left(\frac{2v^2 - 1}{v^3} \right) = \ln \left(\frac{2(y/x)^2 - 1}{(y/x)^3} \right) = \ln \left(\frac{2xy^2 - x^3}{y^3} \right). \end{aligned}$$

So the general solution is $y^3 = 2y^2 - x^2$.

Solution 1.40

The equation can be rewritten as

$$\frac{dy}{dx} = \frac{y}{x} \left(1 + \ln \left(\frac{y}{x} \right) \right),$$

where the right-hand side is clearly a function of y/x .

Now put $y = xv$, so $y' = xv' + v$ and the equation becomes $x \frac{dv}{dx} + v = v(1 + \ln v)$,

which reduces to $x \frac{dv}{dx} = v \ln v$. This equation is separable and gives

$$\int \frac{dx}{x} = \int \frac{dv}{v \ln v}.$$

The integral on the right can be evaluated using the substitution $u = \ln v$, and we obtain $\ln x = \ln(\ln v) + \ln C$. Thus $x = C \ln v$ or $v = e^{Bx}$ (changing the constant). Finally, replacing v by y/x , we have the solution $y = xe^{Bx}$.

Solution 1.41

First we solve the quadratic equation

$$x \left(\frac{dy}{dx} \right)^2 - 2y \frac{dy}{dx} - x = 0 \quad (1.14)$$

to obtain the two possible values

$$\frac{dy}{dx} = \frac{y \pm \sqrt{x^2 + y^2}}{x}.$$

This can be rewritten as

$$\frac{dy}{dx} = \frac{y}{x} \pm \sqrt{1 + \left(\frac{y}{x} \right)^2},$$

so it is homogeneous. Putting $y = xv$ gives

$$x \frac{dv}{dx} + v = v \pm \sqrt{1 + v^2},$$

so

$$\int \frac{dx}{x} = \pm \int \frac{dv}{\sqrt{1 + v^2}}.$$

Thus $\ln x + \ln c = \pm \sinh^{-1} v$ or $v = \pm \sinh(\ln(cx))$. Hence

$$\frac{y}{x} = \pm \frac{e^{\ln(cx)} - e^{-\ln(cx)}}{2},$$

which gives

$$2cy = \pm (c^2 x^2 - 1).$$

It is interesting to note that equation (1.14) also has the solution $y = \pm ix$, which cannot be obtained from the above solution. This is because it is a *non-linear* differential equation.

Solution 1.42

The complementary function (i.e. the solution of the homogeneous equation) is found by solving the quadratic equation

$$m^2 - 7m + 12 = 0.$$

The roots of this equation are $m_1 = 3$ and $m_2 = 4$, so the complementary function is $y_c = Ae^{3x} + Be^{4x}$.

In order to find a particular integral, we try a solution of the form $y = ax^2 + bx + c$, and when we substitute this into the differential equation and equate coefficients, we find that $a = 1$, $b = 2$ and $c = \frac{1}{12}$.

Thus the required general solution is

$$y = Ae^{3x} + Be^{4x} + x^2 + 2x + \frac{1}{12}.$$

Solution 1.43

(a) Let $\frac{dy}{dx}$ be the gradient at a fixed point (x, y) on the curve, with $x \geq 0$ and $y \geq 0$. Then the equation of the tangent line at this point is

$$Y - y = \frac{dy}{dx}(X - x),$$

where X and Y vary along the line. When $X = 0$, we have

$$Y = y - x \frac{dy}{dx},$$

and the given condition then becomes

$$x \sqrt{\left(\frac{dy}{dx} \right)^2 + 1} = y - x \frac{dy}{dx}.$$

This equation can be rearranged to give

$$x^2 = y^2 - 2xy \frac{dy}{dx},$$

and then

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

This equation is homogeneous, so we substitute $y = xv$ to obtain

$$x \frac{dv}{dx} + v = \frac{v^2 - 1}{2v}.$$

Then

$$x \frac{dv}{dx} = -\frac{1 + v^2}{2v},$$

so

$$-\int \frac{dx}{x} = \int \frac{2v}{1 + v^2} dv.$$

Integrating, we obtain $-\ln x + \ln(2C) = \ln(1 + v^2)$, which gives $x^2 + y^2 = 2Cx$ or

$$(x - C)^2 + y^2 = C^2$$

(a semi-circle of arbitrary radius C with centre at $(C, 0)$).

- (b) In order to find the orthogonal family of curves, we must solve the equation

$$\frac{dy}{dx} = -\frac{2xy}{y^2 - x^2}.$$

Substituting $y = xv$ gives

$$x \frac{dv}{dx} + v = -\frac{2v}{v^2 - 1},$$

so

$$x \frac{dv}{dx} = \frac{v(1 + v^2)}{1 - v^2}.$$

Integrating gives

$$\int \frac{dx}{x} = \int \frac{1 - v^2}{v(1 + v^2)} dv = \int \left(\frac{1}{v} - \frac{2v}{1 + v^2} \right) dv,$$

so $\ln x - \ln(2A) = \ln v - \ln(1 + v^2)$, which simplifies to $\frac{x}{2A} = \frac{v}{1 + v^2}$ and therefore

$$x^2 + (y - A)^2 = A^2$$

(a semi-circle of radius A with centre at $(0, A)$).

The two families of curves are shown in Figure 1.12.

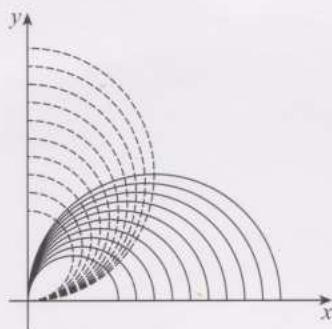


Figure 1.12 Each curve of the second family is orthogonal to each member of the first family

Two lines with gradients T and $-1/T$ will always be orthogonal.

Solution 1.44

Suppose that at time t there are $z(t)$ grams of substance C, which must have been formed from $2z(t)/3$ grams of A and $z(t)/3$ grams of B. Then

$$\frac{dz}{dt} = k \left(200 - \frac{2z}{3} \right) \left(100 - \frac{z}{3} \right),$$

where k is a constant which dictates the reaction rate. This can be rearranged to give

$$\frac{2k}{9} \int dt = \frac{2kt}{9} = \int \frac{dz}{(z-300)^2} = \frac{1}{300-z} + c.$$

Since $z(0) = 0$, we see that $c = -1/300$, thus

$$\frac{2kt}{9} = \frac{1}{300-z} - \frac{1}{300}.$$

We can deduce the reaction rate k from the fact that $z(30) = 100$, which gives

$$\frac{60k}{9} = \frac{1}{200} - \frac{1}{300} = \frac{1}{600},$$

so $k = 1/4000$. It follows that

$$z(t) = \frac{300t}{t+60}.$$



CHAPTER 2

Multivariable and vector calculus

2.1 Introduction

As in the previous chapter, our intention here is to provide some of the basic tools of applied mathematics, and the emphasis is on methods rather than justification and proof.

Some of the material may be familiar, so you can treat it as revision, but there is much that you may find new. The reasons for introducing some of the techniques may not be entirely apparent at this stage, but their relevance will become evident as the course develops. For the moment you should concentrate on acquiring the skills that will be needed later, and attempt as many of the exercises as possible.

Some ideas, such as continuity, are included because they are an essential component of every mathematician's background, but we shall not ask detailed questions on their use. Other notions, such as partial differentiation, will be used throughout the course and you will need to gain some expertise before you move on to later chapters. Generally, you will be able to judge the importance of the topics by the number of exercises we ask you to attempt. However, the Gauss divergence theorem is an exception and will be of crucial importance later.

Subsections 2.2.1 and 2.2.2 are mainly background reading. Subsections 2.2.3 and 2.2.4 discuss various techniques that will be essential later on.

Subsections 2.2.5 to 2.2.8 discuss material which may be familiar, but they lead on to ideas that we shall pursue in Block III.

Subsection 2.3.1 covers material that should be familiar, and, while the remaining subsections discuss techniques that are of universal application, they lead on to the Gauss divergence theorem, which is the most important result of this chapter.

Section 2.4 introduces techniques that will be used throughout the course.

2.2 Multivariable calculus

Functions of two variables

Most of the techniques that we discuss in this section can be extended to functions of three or more variables, but we concentrate on functions of two variables for two reasons. First, while the complexity of the results generally increases with the number of independent variables, functions of two variables often provide a useful insight into the more general cases. Secondly, functions of two variables are relatively easy to visualise because they can often be represented graphically in a three-dimensional coordinate system.

A real function of two real variables $z = F(x, y)$ is a mapping from a set of points D in the Cartesian plane to the real numbers. The set D is known as the domain of the function, and each point (x, y) of D maps to a unique value z . The variables x and y are known as the independent variables, and z is the dependent variable.

Consider the real function F defined by $F(x, y) = \sqrt{1 - (x^2 + y^2)}$. For $F(x, y)$ to be real, we require $x^2 + y^2 \leq 1$, so we take the disc defined by this inequality to be the domain of F .

Remember that the square root symbol indicates the positive square root.

Here, and generally, we assume that the domain of a function is the largest set on which the definition is sensible.

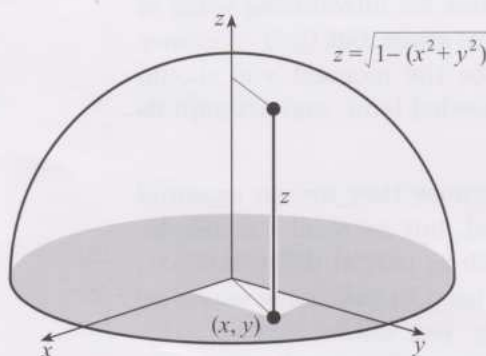


Figure 2.1 The hemisphere $z = \sqrt{1 - (x^2 + y^2)}$

If we put $z = \sqrt{1 - (x^2 + y^2)}$ and let (x, y, z) correspond to a point in a three-dimensional Cartesian diagram, then the function F can be represented as a hemisphere, as shown in Figure 2.1.

Generally, a function $w = F(x, y)$ can be represented as a surface, and $(x, y, F(x, y))$ represents an arbitrary point on the surface.

We use either (x, y) or $(x, y, 0)$ when referring to points that lie in the (x, y) -plane.

2.2.1 Level curves

It is not always easy to visualise a surface in three dimensions, and there is an alternative pictorial representation of a function of two variables that is sometimes helpful. On an ordnance survey map you may have seen lines connecting points of equal height above sea level. Such lines are known as **contours**, and they give some indication of the shape of the surface. The same idea can help us to understand the behaviour of functions.

If we are given a function $z = F(x, y)$, then we may be able to plot the contours or **level curves** corresponding to the equations $F(x, y) = \lambda$ for various values of the constant λ .

For the function $F(x, y) = \sqrt{1 - (x^2 + y^2)}$ we consider the curves

$$\sqrt{1 - (x^2 + y^2)} = \lambda, \quad \text{where } 0 \leq \lambda \leq 1.$$

In this case it is convenient to rearrange the equation into the form

$$x^2 + y^2 = 1 - \lambda^2,$$

and we can then see that the level curves are circles of radius $\sqrt{1 - \lambda^2}$, as shown in Figure 2.2. Such a diagram can often give some insight into the behaviour of an unfamiliar function.

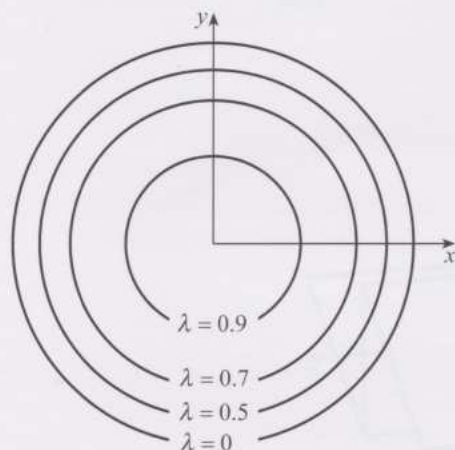


Figure 2.2 The level curves $x^2 + y^2 = 1 - \lambda^2$

Exercise 2.1

Sketch the level curves of the functions $F(x, y) = 3 - 2x + y^2$ and $G(x, y) = y/x^2$.

2.2.2 Continuity

Perhaps the best way to appreciate the meaning of continuity is to examine a function that fails to be continuous; we begin with a function of one variable.

Example 2.1

Sketch the graph of the function $f(x) = \frac{x}{|x|}$.

Solution

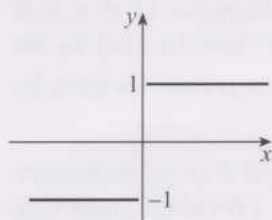


Figure 2.3 $f(x) = 1$ if $x > 0$ and $f(x) = -1$ if $x < 0$

The function is defined for all real values of x except $x = 0$, with

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

The graph is shown in Figure 2.3 with a break at the origin. The essential point here is that *whatever* value we assign to $f(0)$, the resulting function will be discontinuous at the origin. ■

You saw in Chapter 1 that we define a function $f(x)$ of one variable to be continuous at $x = a$ if $\lim_{x \rightarrow a} f(x) = f(a)$. Clearly, in the above example $\lim_{x \rightarrow 0} f(x)$ does not exist because $f(x)$ approaches -1 when x approaches 0 from below, and $f(x)$ approaches 1 when x approaches 0 from above.

Now we move on to discuss functions of two variables.

Example 2.2

Sketch the surface that represents the function $F(x, y) = \frac{xy}{|xy|}$.

Solution

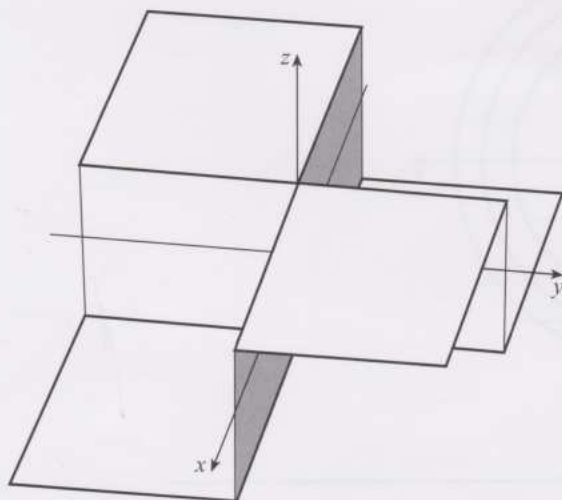


Figure 2.4 The function $F(x, y) = \frac{xy}{|xy|}$

We see that $F(x, y) = 1$ when x and y have the same sign, and $F(x, y) = -1$ when they do not. The corresponding surface is shown in Figure 2.4, and it is clear that the function behaves badly on the axes. First, the function is not defined on the x - and y -axes, but, even if it were to be defined there, the sudden changes in value from -1 to 1 would ensure that the function is discontinuous. ■

A geometric description of continuity is a useful aid to understanding, but it cannot be extended easily to functions of more than two variables. A slightly different approach, based on comparing the values of the function at neighbouring points in the domain, is more productive.

For a function of one variable, we define f to be continuous at $x = a$ if $\lim_{x \rightarrow a} f(x) = f(a)$. In other words, we can make $f(x)$ as close to $f(a)$ as we please by choosing x sufficiently close to a . It is this notion that we wish to extend to functions of two variables.

We say that a function F is continuous at a point (a, b) if $F(x, y)$ converges to $F(a, b)$ as (x, y) approaches (a, b) from any direction. (We shall make this statement a little more precise when we discuss functions of many variables later.)

In this course the difficulties that arise from discontinuous functions are not our main concern, so you may assume that any given function is continuous unless we say otherwise.

Many of the results that come later in the course are crucially dependent on the fact that the relevant functions are continuous, but we shall be avoiding these complications.

2.2.3 Partial derivatives

In addition to continuity, we often require that a function should be ‘smooth’, which in intuitive terms means that there are no sudden dents or sharp corners on the surface that represents the function. This is equivalent to the requirement that it is possible to construct a flat surface, known as the **tangent plane**, that touches the original surface at the point in question. The existence of such a tangent plane is intimately connected to the existence of the **partial derivatives** of the function, so first we introduce the notion of partial derivatives and then we use them to define the tangent plane.

The partial derivative of a function $F(x, y)$ with respect to x is obtained by differentiating the function with respect to x while keeping the variable y fixed, and similarly for the partial derivative with respect to y . These partial derivatives are often denoted by

$$\frac{\partial F}{\partial x} \quad \text{and} \quad \frac{\partial F}{\partial y},$$

although alternative notations can be useful, e.g. $F_x(x, y)$ and $F_y(x, y)$, or just F_x and F_y .

The partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$, when evaluated at $x = a$ and $y = b$, represent the slope of the surface $z = F(x, y)$ at the point $(a, b, F(a, b))$ in the x and y directions, respectively.

Example 2.3

Given $F(x, y) = x^2 + y^3$, calculate $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$, and hence deduce the slope in the x and y directions at the point $(3, 2)$.

Solution

To calculate $\frac{\partial F}{\partial x}$, we differentiate with respect to x keeping y constant:

$$\frac{\partial F}{\partial x} = 2x.$$

To calculate $\frac{\partial F}{\partial y}$, we differentiate with respect to y keeping x constant:

$$\frac{\partial F}{\partial y} = 3y^2.$$

Alternatively, we might have written $z = x^2 + y^3$, in which case $\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 3y^2$.

Hence the slopes in the x and y directions at the point $(3, 2)$ are $F_x(3, 2) = 6$ and $F_y(3, 2) = 12$, respectively. ■

Exercise 2.2

Given $F(x, y) = x^2y^3 + \sin(xe^y)$, calculate $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$.

Example 2.4

Given an equation such as $x^2 + y^2 - z^2 = 1$, we need to clarify which are the dependent and independent variables.

- (a) Supposing that z is the dependent variable, while x and y are the independent variables, evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- (b) Alternatively, supposing that y is the dependent variable, while x and z are independent, evaluate $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial z}$.

Solution

- (a) Rearranging the equation, we obtain $z = \pm\sqrt{x^2 + y^2 - 1}$, which is real if $x^2 + y^2 \geq 1$. Choosing the positive square root, and differentiating first with respect to x and then with respect to y , we obtain

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{(x^2 + y^2) - 1}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{(x^2 + y^2) - 1}}$$

(with similar results if we choose the negative root).

Alternatively, we may differentiate the equation $x^2 + y^2 - z^2 = 1$ directly. First we differentiate partially with respect to x , and notice that the partial derivatives of x^2 and y^2 with respect to x are just $2x$ and 0 , respectively, while the partial derivative of z^2 (a function of x and y) is $2z \frac{\partial z}{\partial x}$ (keeping y fixed and using the chain rule). Thus we obtain

$$2x - 2z \frac{\partial z}{\partial x} = 0.$$

Similarly, differentiating partially with respect to y , we obtain

$$2y - 2z \frac{\partial z}{\partial y} = 0.$$

Rearranging these equations, we have

$$\frac{\partial z}{\partial x} = \frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{z}, \quad \text{where } z \neq 0.$$

- (b) Differentiating the equation $x^2 + y^2 - z^2 = 1$ partially with respect to x and z now gives

$$2x + 2y \frac{\partial y}{\partial x} = 0 \quad \text{and} \quad 2y \frac{\partial y}{\partial z} - 2z = 0,$$

and rearranging gives

$$\frac{\partial y}{\partial x} = -\frac{x}{y} \quad \text{and} \quad \frac{\partial y}{\partial z} = \frac{z}{y}, \quad \text{where } y \neq 0. \quad \blacksquare$$

Example 2.5

Show that $\frac{d}{dt} \left(\int_0^1 \frac{dx}{(x+t)^2} \right) = \int_0^1 \frac{\partial}{\partial t} \left(\frac{1}{(x+t)^2} \right) dx$.

Solution

We have

$$\frac{d}{dt} \left(\int_0^1 \frac{dx}{(x+t)^2} \right) = \frac{d}{dt} \left[-\frac{1}{x+t} \right]_0^1 = \frac{d}{dt} \left(\frac{1}{t} - \frac{1}{1+t} \right) = \frac{1}{(1+t)^2} - \frac{1}{t^2}$$

and

$$\int_0^1 \frac{\partial}{\partial t} \left(\frac{1}{(x+t)^2} \right) dx = \int_0^1 \frac{-2dx}{(x+t)^3} = \left[\frac{1}{(x+t)^2} \right]_0^1 = \frac{1}{(1+t)^2} - \frac{1}{t^2}. \quad \blacksquare$$

Note that we use partial derivatives (∂) to differentiate the integrand, and total derivatives (d) to differentiate the integral. This is because the integrand is a function of x and t , while the integral is just a function of t .

Generally, we shall assume that we are justified in ‘differentiating through the integral sign’, but if necessary we may appeal to the following result.

Leibniz’s rule

Let $F(x, t)$ be a continuous function with a continuous partial derivative $\partial F/\partial t$ on the rectangular region $a \leq x \leq b, t_1 \leq t \leq t_2$ of the (x, t) -plane. Then, for $t_1 < t < t_2$,

$$\frac{d}{dt} \int_a^b F(x, t) dx = \int_a^b \frac{\partial F}{\partial t} dx.$$

This result will be used in Block III.

Exercise 2.3

Evaluate $\int_0^1 \sin(ax) dx$, then use Leibniz’s rule to evaluate $\int_0^1 x \cos(ax) dx$.
(It is not necessary to verify the conditions for Leibniz’s rule.)

2.2.4 Linear approximation

It is often useful to obtain an approximation to a given function near a given point. For a function of one variable, $y = f(x)$ say, we may use a linear approximation $f(x) \simeq f(a) + f'(a)(x - a)$, which is equivalent to approximating the function by the tangent line at $x = a$.

For a function of two variables we can obtain a similar linear estimate in terms of the partial derivatives, and in geometric terms this estimate can be interpreted as approximation by the tangent plane.

The essential idea is that we find a plane that touches the corresponding surface at a given point, and then we approximate values on the surface by values on this nearby plane. To make the ideas more precise, we need the equation of a plane.

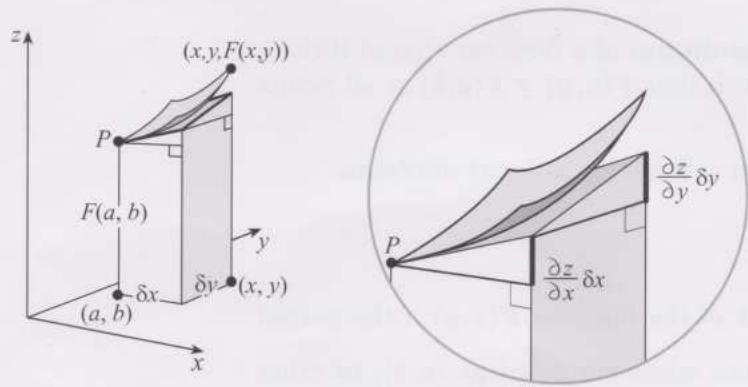


Figure 2.5 The surface representing F near the point P

In Figure 2.5 the surface corresponding to the function $z = F(x, y)$ passes through the point P with coordinates $(a, b, F(a, b))$, and our aim is to estimate the value of $F(x, y)$ when (x, y) is close to (a, b) .

The equation of a plane

The equation $z = A + B(x - a) + C(y - b)$ represents a plane passing through the point (a, b, A) . Differentiating partially with respect to x and then with respect to y , we obtain

$$\frac{\partial z}{\partial x} = B \quad \text{and} \quad \frac{\partial z}{\partial y} = C.$$

In other words, the slope of the plane in the x direction is B , while the slope in the y direction is C .

The tangent plane to the surface $z = F(x, y)$ at the point $(a, b, F(a, b))$ is defined as having the same slope as the surface in the x and y directions, so its equation is

$$z = F(a, b) + (x - a)\frac{\partial F}{\partial x} + (y - b)\frac{\partial F}{\partial y},$$

where the partial derivatives are evaluated at (a, b) . Thus our linear approximation for the given function is

$$F(x, y) \simeq F(a, b) + (x - a)\frac{\partial F}{\partial x} + (y - b)\frac{\partial F}{\partial y}.$$

This result is often written in terms of the delta notation for small increments, in which case it becomes

$$\delta F \simeq \frac{\partial F}{\partial x} \delta x + \frac{\partial F}{\partial y} \delta y, \quad (2.1)$$

where $\delta F = F(x, y) - F(a, b)$, $\delta x = x - a$ and $\delta y = y - b$. Equation (2.1) tells us approximately how much the function F changes when we move from a point (a, b) to a neighbouring point $(a + \delta x, b + \delta y)$. This approximate change in the function is depicted in the inset of Figure 2.5.

2.2.5 Local extrema

You will probably recall that the equation $(x - a)^2 + (y - b)^2 = \delta^2$ defines a circle of radius δ with its centre at (a, b) . The interior of this circle is defined by the inequality $(x - a)^2 + (y - b)^2 < \delta^2$.

A point (a, b) is said to be a **local maximum** of a function $F(x, y)$ if there is a region $(x - a)^2 + (y - b)^2 < \delta^2$ such that $F(x, y) \leq F(a, b)$ at all points (x, y) in the region.

A point (a, b) is said to be a **local minimum** of a function $F(x, y)$ if there is a region $(x - a)^2 + (y - b)^2 < \delta^2$ such that $F(x, y) \geq F(a, b)$ at all points (x, y) in the region.

Local maxima and minima are known collectively as **local extrema**.

Stationary points

A point (a, b) is a **stationary point** of the function $F(x, y)$ if the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are both zero when evaluated at (a, b) ; in other words, $F_x(a, b) = F_y(a, b) = 0$.

Stationary points of functions of two variables play much the same role as such points for functions of one variable. This stems from the fact that it is possible to show that (for a well-behaved function) every extremum must be

a stationary point. Thus, if we wish to find a local maximum or minimum of a given function, we may restrict our search to its stationary points. On the other hand, not all stationary points are local maxima or minima. A stationary point that is not an extremum is called a **saddle point**.

Figure 2.6 shows examples of typical stationary points.

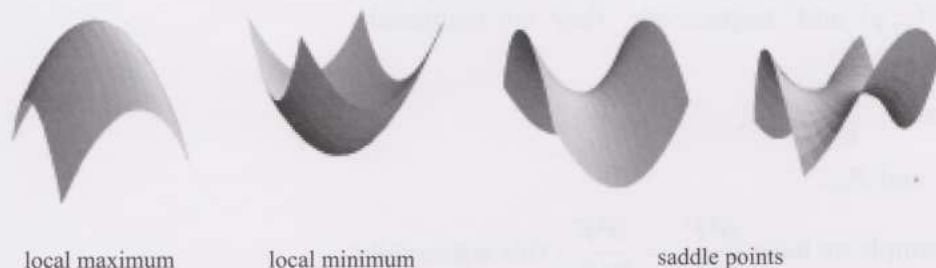


Figure 2.6 Typical behaviour of a function near a stationary point

Exercise 2.4

- (a) Find the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ of the function

$$F(x, y) = e^{xy} + x^2 + \sin y.$$

- (b) Given $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$, and regarding z as the dependent variable, calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

- (c) Given $\cos(xyz) = 1$, and regarding z as the dependent variable, calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

- (d) (i) Show that the surface $z = F(x, y) = 1 + (x - 2y + 1)(y - 2x + 1)$ passes through the point $(1, 2, -1)$, and find the tangent plane at that point.

- (ii) Show that $F(x, y)$ has a stationary point at $(1, 1)$. Show that the tangent plane at the point $(1, 1, 1)$ is parallel to the (x, y) -plane. Putting $s = x - 1$ and $t = y - 1$, or otherwise, find an expression for $F(x, y) - F(1, 1)$ in terms of powers of $x - 1$ and $y - 1$.

- (iii) The lines $y = 2 - x$ and $y = x$ intersect at the point $(1, 1)$. What can you say about the sign of $F(x, y) - F(1, 1)$ on the line $y = 2 - x$ and on the line $y = x$? What do you conclude about the stationary point at $(1, 1)$?

- (e) Show that the function $F(x, y) = (x^2 + y^2)e^x$ has two stationary points, and that the stationary point at the origin is an extremum.

2.2.6 Higher-order partial derivatives

From Exercise 2.4, you may have observed that the partial derivatives of a function $F(x, y)$ are themselves functions of two variables. This means that they too can be differentiated partially with respect to x and y .

For example, if $F(x, y) = x^2 \sin y$, then

$$\frac{\partial F}{\partial x} = 2x \sin y \quad \text{and} \quad \frac{\partial F}{\partial y} = x^2 \cos y.$$

Differentiating again,

$$\begin{aligned}\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial x} \right) &= 2 \sin y, & \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial x} \right) &= 2x \cos y, \\ \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y} \right) &= 2x \cos y & \text{and} & \quad \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial y} \right) = -x^2 \sin y.\end{aligned}$$

These are known as the *second-order partial derivatives* or *second partial derivatives* of the function $F(x, y)$ and, respectively, they are commonly abbreviated to

$$\frac{\partial^2 F}{\partial x^2}, \quad \frac{\partial^2 F}{\partial y \partial x}, \quad \frac{\partial^2 F}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 F}{\partial y^2},$$

or alternatively F_{xx} , F_{yx} , F_{xy} and F_{yy} .

It is no accident that in this example we have $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$; this is generally true provided that the function $F(x, y)$ is sufficiently smooth for the second-order partial derivatives to exist and to be continuous. (This result is often known as the **mixed derivative theorem**; we shall not prove it here.)

It is, of course, possible to take still higher derivatives, e.g. F_{xxx} , F_{yxx} . The mixed derivative theorem then states that the order in which these derivatives occur does not matter; so, for example, $F_{yxx} = F_{xyx} = F_{xxy} = F_{xxxy}$.

Taylor approximations

It is often the case that the linear approximation to a function is not sufficiently accurate to provide the information that we require; we may need a higher-order approximation. Suppose that near $(0, 0)$ it is possible to approximate the function $F(x, y)$ by a quadratic expression so that

$$F(x, y) \simeq A + (Bx + Cy) + (Px^2 + 2Qxy + Ry^2),$$

where A , B , C , P , Q and R are constants.

This is known as a quadratic approximation for $F(x, y)$ near $(0, 0)$.

More precisely, we suppose that

$$\begin{aligned}F(x, y) &= A + (Bx + Cy) + (Px^2 + 2Qxy + Ry^2) \\ &\quad + (\text{higher powers of } x \text{ and } y).\end{aligned}$$

Putting $x = y = 0$ on both sides of this equation gives $A = F(0, 0)$.

Differentiating both sides of the equation partially with respect to x gives

$$\frac{\partial F}{\partial x} = B + (2Px + 2Qy) + (\text{higher-order terms});$$

then putting $x = y = 0$ gives $B = F_x(0, 0)$. Similarly, differentiating partially with respect to y and putting $x = y = 0$ gives $C = F_y(0, 0)$.

Differentiating both sides of the approximation twice with respect to x gives

$$\frac{\partial^2 F}{\partial x^2} = 2P + (\text{higher-order terms});$$

then putting $x = y = 0$ gives $P = \frac{1}{2}F_{xx}(0, 0)$. Similarly, $Q = \frac{1}{2}F_{xy}(0, 0)$ and $R = \frac{1}{2}F_{yy}(0, 0)$.

Thus, for a function $F(x, y)$ that is sufficiently smooth near $(0, 0)$, the quadratic approximation to $F(x, y)$ near $(0, 0)$ becomes

$$\begin{aligned}F(x, y) &\simeq F(0, 0) + F_x(0, 0)x + F_y(0, 0)y \\ &\quad + \frac{1}{2} (F_{xx}(0, 0)x^2 + 2F_{xy}(0, 0)xy + F_{yy}(0, 0)y^2). \quad (2.2)\end{aligned}$$

This is known as the **Taylor approximation of order two** for $F(x, y)$ near $(0, 0)$.

(Notice that the linear terms in this expression are identical to the linear approximation that we discussed earlier.)

It is possible to extend this process and obtain an infinite series for $F(x, y)$ in ascending powers of x and y . Such a series is known as the **Taylor series** for a function of two variables.

In this course we shall not generally require approximations of order higher than two.

Example 2.6

Find the Taylor approximation of order two for $F(x, y) = (x + 2y + 1)e^{xy}$ near $(0, 0)$.

Solution

We have

$$F_x = e^{xy} + (x + 2y + 1)ye^{xy} \quad \text{and} \quad F_y = 2e^{xy} + (x + 2y + 1)xe^{xy}.$$

Then

$$\begin{aligned} F_{xx} &= (2 + y + xy + 2y^2)ye^{xy}, \\ F_{xy} &= (1 + 2x + 4y + x^2y + xy + 2xy^2)e^{xy}, \\ F_{yy} &= (4 + x + 2xy + x^2)xe^{xy}. \end{aligned}$$

Thus $F(0, 0) = 1$, $F_x(0, 0) = 1$, $F_y(0, 0) = 2$, $F_{xx}(0, 0) = 0$, $F_{xy}(0, 0) = 1$ and $F_{yy}(0, 0) = 0$. Substituting these values into the general expression (2.2), the Taylor approximation is

$$\begin{aligned} (x + 2y + 1)e^{xy} &\simeq 1 + (x + 2y) + \frac{1}{2}(2xy) \\ &= 1 + x + 2y + xy. \end{aligned}$$

Alternatively, we could obtain the same result more easily by noticing that $e^{xy} = 1 + \frac{1}{1!}(xy) + \frac{1}{2!}(xy)^2 + \dots$, so

$$\begin{aligned} (x + 2y + 1)e^{xy} &\simeq (x + 2y + 1) \left(1 + xy + \frac{1}{2!}(xy)^2 + \dots \right) \\ &= 1 + x + 2y + xy + (\text{higher powers of } x \text{ and } y). \quad \blacksquare \end{aligned}$$

A similar argument applies to approximations near an arbitrary point (a, b) . For a function $F(x, y)$ that is sufficiently smooth near (a, b) , the *Taylor approximation of order two* for $F(x, y)$ near (a, b) is

$$\begin{aligned} F(x, y) &\simeq F(a, b) + F_x(a, b)(x - a) + F_y(a, b)(y - b) \\ &\quad + \frac{1}{2} (F_{xx}(a, b)(x - a)^2 + 2F_{xy}(a, b)(x - a)(y - b) \\ &\quad + F_{yy}(a, b)(y - b)^2). \end{aligned}$$

(Again, the linear terms in this expression are identical to the linear approximation to $F(x, y)$ near (a, b) .)

The corresponding infinite series in powers of $x - a$ and $y - b$ is known as the Taylor series of F at (a, b) .

We shall not require terms of the Taylor series beyond those corresponding to the Taylor approximation of order two. However, if we let $\delta x = x - a$, $\delta y = y - b$, $p_0 = (a, b)$, $\delta p = (\delta x, \delta y)$, and define an operator $U = \delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y}$, then we can write the Taylor series at (a, b) in the neat form

$$F(p_0 + \delta p) = F(p_0) + \sum_{k=1}^{\infty} \frac{1}{k!} U^k F$$

(where all the partial derivatives are evaluated at p_0). The first three terms of this series give the Taylor approximation of order two for F near (a, b) .

The classification of stationary points

Suppose that $F(x, y)$ has a stationary point at the origin, so that the first-order partial derivatives are zero at $(0, 0)$. Then the Taylor approximation of order two for $F(x, y)$ near $(0, 0)$ becomes

$$F(x, y) \simeq F(0, 0) + \frac{1}{2} (F_{xx}(0, 0)x^2 + 2F_{xy}(0, 0)xy + F_{yy}(0, 0)y^2).$$

Near an extremum at $(0, 0)$, it is the sign of $F(x, y) - F(0, 0)$ that determines if the point is a local maximum or a local minimum, so we are led to consider the sign of expressions of the form

$$Px^2 + 2Qxy + Ry^2$$

(where the constants P , Q and R are not all zero). It turns out that the sign of this expression is dependent on the sign of $PR - Q^2$.

Suppose that $PR - Q^2 > 0$. Then P cannot be zero, so we may write

$$Px^2 + 2Qxy + Ry^2 = \frac{(Px + Qy)^2 + (PR - Q^2)y^2}{P},$$

where the numerator is positive (except at the origin).

Thus, at points other than the origin, $Px^2 + 2Qxy + Ry^2$ takes the same sign as P .

Now putting $P = F_{xx}(0, 0)$, $Q = F_{xy}(0, 0)$ and $R = F_{yy}(0, 0)$, we see that the function $F(x, y)$ has an extremum at the origin if

$$F_x(0, 0) = F_y(0, 0) = 0 \quad \text{and} \quad PR - Q^2 > 0,$$

and this extremum is a local maximum or minimum according as P (or R) is negative or positive, respectively.

The above result can be easily extended to a function $F(x, y)$ that has a stationary point at a point (a, b) other than the origin. In this case we need only observe that the function $F(x + a, y + b)$ has a stationary point at the origin, and, applying the previous result, we obtain the following.

A test for extrema

The function $F(x, y)$ has an extremum at the point (a, b) if

$$F_x(a, b) = F_y(a, b) = 0 \quad \text{and} \quad PR - Q^2 > 0,$$

where $P = F_{xx}(a, b)$, $Q = F_{xy}(a, b)$ and $R = F_{yy}(a, b)$.

This extremum is a local maximum or minimum according as P (or R) is negative or positive, respectively.

If $PR - Q^2 < 0$, it can be shown that $F(x, y)$ does not have an extremum at the point (a, b) .

A stationary point that is not an extremum is known as a saddle point.

Near an extremum at (a, b) , $F(x, y) - F(a, b)$ does not change sign, but near a saddle point there are always points for which $F(x, y) - F(a, b)$ is positive and points for which it is negative.

If $PR - Q^2 = 0$, then the above test for an extremum fails. The point in question may or may not be an extremum, and this may be determined by examining a higher-order approximation, although we shall not discuss the general process here.

Example 2.7

Find and classify the stationary points of the function $w = x^3 + y^3 + 3xy$.

Solution

We have

$$\frac{\partial w}{\partial x} = 3x^2 + 3y \quad \text{and} \quad \frac{\partial w}{\partial y} = 3y^2 + 3x,$$

so at a stationary point, $x^2 + y = 0$ and $y^2 + x = 0$. Solving this pair of equations, we obtain $x = 0$ and $y = 0$, or $x = -1$ and $y = -1$, giving stationary points at $(0, 0)$ and $(-1, -1)$. We also have

$$\frac{\partial^2 w}{\partial x^2} = 6x, \quad \frac{\partial^2 w}{\partial x \partial y} = 3 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2} = 6y,$$

so

$$PR - Q^2 = \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 = 36xy - 9.$$

At $(0, 0)$ we have $PR - Q^2 = -9 < 0$, so the stationary point is a saddle point. At $(-1, -1)$ we have $PR - Q^2 = 27 > 0$, so the stationary point is an extremum. Also at this point we have $P = \frac{\partial^2 w}{\partial x^2} = 6x$ evaluated at $(-1, -1)$, giving $P = -6 < 0$, so the extremum is a local maximum. ■

In the functions in the following example, $PR - Q^2 = 0$, so the test fails. However, in these cases, we can use elementary analysis to determine the nature of the stationary point.

Example 2.8

- Given $F(x, y) = x^4 y^4$, show that $PR - Q^2 = 0$ and that the function has a local minimum at $(0, 0)$.
- Given $F(x, y) = x^3 y^3$, show that $PR - Q^2 = 0$ and that the function has a saddle point at $(0, 0)$.

Solution

- Since $F_x(0, 0) = F_y(0, 0) = 0$, $(0, 0)$ is a stationary point. We have

$$\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 = (12x^2 y^4)(12x^4 y^2) - (16x^3 y^3)^2,$$

so at $(0, 0)$ we have $PR - Q^2 = 0$. However, $F(x, y) = x^4 y^4 \geq 0 = F(0, 0)$, so there is a local minimum at $(0, 0)$.

- Since $F_x(0, 0) = F_y(0, 0) = 0$, $(0, 0)$ is a stationary point. We have

$$\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 = (6xy^3)(6x^3 y) - (9x^2 y^2)^2,$$

so at $(0, 0)$ we have $PR - Q^2 = 0$. However, $F(x, y) - F(0, 0) = x^3 y^3$, which is positive when x and y have the same sign, and negative when they have opposite signs. Thus there cannot be a local extremum at the origin, and the point is therefore a saddle point. ■

Exercise 2.5

- Find the Taylor approximation of order two to the function $w = e^{xy}$ near $(1, 2)$.
- Find and classify the stationary points of the function $w = \sin x \cosh y$.

Later we shall discuss partial differential equations, and this is an opportune moment to introduce a solution of such an equation.

Recall that a partial differential equation is any equation involving partial derivatives.

Exercise 2.6

Show that any function of the form $F(x, y) = f(x) + g(y)$, where f and g are differentiable, satisfies the equation $\frac{\partial^2 F}{\partial x \partial y} = 0$.

2.2.7 Functions of many variables

Suppose that we wish to investigate the behaviour of a function F of the n variables $x_1, x_2, \dots, x_n$. It is often convenient to write $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and to regard F as a function of the vector $\mathbf{x}$.

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{a} = (a_1, a_2, \dots, a_n)$. Then we define a function $F(\mathbf{x})$ to be continuous at $\mathbf{a}$ if $\lim_{\mathbf{x} \rightarrow \mathbf{a}} F(\mathbf{x}) = F(\mathbf{a})$, and note that $\mathbf{x}$ may approach $\mathbf{a}$ from any direction.

We shall discuss vectors in the second part of this chapter.

The linear approximation is easily extended to functions of many variables. If $\mathbf{a} = (a_1, a_2, \dots, a_n)$, we write $\delta F = F(\mathbf{x}) - F(\mathbf{a})$ and $\delta x_i = x_i - a_i$ for $i = 1, 2, \dots, n$. We then have

$$\delta F \simeq \frac{\partial F}{\partial x_1} \delta x_1 + \frac{\partial F}{\partial x_2} \delta x_2 + \dots + \frac{\partial F}{\partial x_n} \delta x_n,$$

where all the partial derivatives are evaluated at $\mathbf{a}$.

It is also possible to extend Taylor approximations of higher order to functions of many variables, although, as one might expect, the complexity of the expressions increases with the number of variables and with the order of the approximation.

Exercise 2.7

Given $\mathbf{x} = (x_1, x_2, x_3)$ and

$$F(\mathbf{x}) = A + (B_1 x_1 + B_2 x_2 + B_3 x_3) + (C_{11} x_1^2 + C_{22} x_2^2 + C_{33} x_3^2 + C_{12} x_1 x_2 + C_{23} x_2 x_3 + C_{31} x_3 x_1),$$

calculate the coefficients in this expression in terms of partial derivatives of F .

Taylor approximation of order two

For an arbitrary function $F(\mathbf{x})$ of three variables, the Taylor approximation of order two near $\mathbf{x} = \mathbf{a}$ is

Exercise 2.7 illustrates such an approximation when $\mathbf{a} = \mathbf{0}$.

$$\begin{aligned} F(\mathbf{x}) \simeq & F(\mathbf{a}) + (x_1 - a_1) \frac{\partial F}{\partial x_1} + (x_2 - a_2) \frac{\partial F}{\partial x_2} + (x_3 - a_3) \frac{\partial F}{\partial x_3} \\ & + \frac{1}{2} \left((x_1 - a_1)^2 \frac{\partial^2 F}{\partial x_1^2} + (x_2 - a_2)^2 \frac{\partial^2 F}{\partial x_2^2} + (x_3 - a_3)^2 \frac{\partial^2 F}{\partial x_3^2} \right. \\ & + 2(x_1 - a_1)(x_2 - a_2) \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ & + 2(x_2 - a_2)(x_3 - a_3) \frac{\partial^2 F}{\partial x_2 \partial x_3} \\ & \left. + 2(x_3 - a_3)(x_1 - a_1) \frac{\partial^2 F}{\partial x_3 \partial x_1} \right), \end{aligned}$$

where all the partial derivatives are evaluated at $\mathbf{x} = \mathbf{a}$. It is easy to see how this result can be extended to more than three variables.

Stationary points

The notions of stationary points and extrema can be easily extended to functions of more than two variables.

Near the point $\mathbf{a} = (a_1, a_2, \dots, a_n)$, it is the nature of the difference $\delta F = F(\mathbf{x}) - F(\mathbf{a})$ that characterises the behaviour of the function F . If this difference is positive (or zero) for all $\mathbf{x}$ close to $\mathbf{a}$, then the function will have a local minimum at $\mathbf{a}$; alternatively, if the difference is negative (or zero), then the function will have a local maximum at $\mathbf{a}$. However, if in every neighbourhood of $\mathbf{a}$ there are points at which $\delta F = F(\mathbf{x}) - F(\mathbf{a})$ is positive and points at which it is negative, then there cannot be an extremum at $\mathbf{a}$.

There is an alternative approach to stationary points which will be relevant in later chapters. The idea is to examine the behaviour of a function as we approach a stationary point from a nearby point. In Block III we shall see that this alternative approach can be extended to a situation in which there is no direct analogue of the notion of a partial derivative.

Since we are concerned only with points close to $\mathbf{a}$, it is convenient to write $\mathbf{x} = \mathbf{a} + \varepsilon \boldsymbol{\xi}$, where $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_n)$, with $\xi_1^2 + \xi_2^2 + \dots + \xi_n^2 = 1$, and ε is small. (Essentially, this means that we approach the point $\mathbf{a}$ along the straight line joining $\mathbf{a}$ to $\mathbf{a} + \varepsilon \boldsymbol{\xi}$.) We expect δF to be small when ε is small. We might expect that $\delta F = F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a})$ is approximately proportional to ε , or, more specifically, that

$$\frac{F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a})}{\varepsilon}$$

is bounded (i.e. not infinite) when ε is small. For well-behaved functions, and for an arbitrary point $\mathbf{a}$, this is generally the case, and we have

$$F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a}) \simeq \varepsilon \left(\frac{\partial F}{\partial x_1} \xi_1 + \frac{\partial F}{\partial x_2} \xi_2 + \dots + \frac{\partial F}{\partial x_n} \xi_n \right) = O(\varepsilon).$$

Example 2.9

Given $F(\mathbf{x}) = x_1 x_2 + x_2 x_3 + x_3 x_1$ and $\mathbf{a} = (1, -1, 2)$, find $F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a})$ and show that $F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a}) = O(\varepsilon)$ for all $\boldsymbol{\xi}$.

Solution

We have

$$\begin{aligned} F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) &= (1 + \varepsilon \xi_1)(-1 + \varepsilon \xi_2) + (-1 + \varepsilon \xi_2)(2 + \varepsilon \xi_3) \\ &\quad + (2 + \varepsilon \xi_3)(1 + \varepsilon \xi_1) \\ &= -1 + \varepsilon(-\xi_1 + \xi_2 - \xi_3 + 2\xi_2 + \xi_3 + 2\xi_1) \\ &\quad + \varepsilon^2(\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_1) \\ &= -1 + \varepsilon(\xi_1 + 3\xi_2) + \varepsilon^2(\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_1) \end{aligned}$$

and $F(\mathbf{a}) = -1$, so

$$F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a}) = \varepsilon(\xi_1 + 3\xi_2) + \varepsilon^2(\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_1) = O(\varepsilon). \quad \blacksquare$$

The previous example illustrates what we expect to happen for an arbitrarily chosen point $\mathbf{a}$. If ε is small, then ε^2 is smaller, so the difference $F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a})$ is approximately proportional to ε .

However, for a given function F , there may be exceptional points $\mathbf{a}$ for which

$$F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a}) = O(\varepsilon^2) \quad \text{for all } \boldsymbol{\xi}$$

(in other words, δF is smaller than we might expect). These are, of course, the points for which

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \cdots = \frac{\partial F}{\partial x_n} = 0,$$

so that the linear approximation for δF is zero. These exceptional points are in fact the stationary points of F .

Example 2.10

Given $F(\mathbf{x}) = x_1x_2 + x_2x_3 + x_3x_1$ and $\mathbf{a} = (0, 0, 0)$, find $F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a})$ and show that $F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a}) = O(\varepsilon^2)$, so F has a stationary point at the origin.

Solution

In this case we have

$$F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a}) = \varepsilon^2(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1) = O(\varepsilon^2),$$

and we conclude that the point $\mathbf{a} = (0, 0, 0)$ is a stationary point.

We could, of course, have reached the same conclusion by calculating the partial derivatives

$$\frac{\partial F}{\partial x_1} = x_2 + x_3, \quad \frac{\partial F}{\partial x_2} = x_3 + x_1 \quad \text{and} \quad \frac{\partial F}{\partial x_3} = x_1 + x_2.$$

We can see that $\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial x_3} = 0$ at $(0, 0, 0)$, which confirms that it is a stationary point. ■

We could, therefore, have defined the stationary points of a function F to be those points $\mathbf{a}$ for which $F(\mathbf{a} + \varepsilon \boldsymbol{\xi}) - F(\mathbf{a}) = O(\varepsilon^2)$, and indeed it is this formulation that is most easily extended to the cases we shall consider later in the course.

Generally, the classification of stationary points (into maxima, minima and saddle points) involves the determination of the sign of a quadratic expression in many variables, but we shall not discuss that process for functions of more than two variables.

2.2.8 The chain rule

Often we encounter situations where we wish to transfer between coordinate systems. For example, we may find it convenient to transfer from Cartesian coordinates (x, y) to polar coordinates (r, θ) , in which case we may wish to represent partial derivatives with respect to x and y in terms of partial derivatives with respect to r and θ (or vice versa). The device which makes this possible is known as the *chain rule*.

Suppose that we are given a variable w which is dependent on two variables, x and y . Then the linear approximation gives

$$\delta w \simeq \frac{\partial w}{\partial x} \delta x + \frac{\partial w}{\partial y} \delta y, \quad (2.3)$$

where the accuracy of the approximation increases as δx and δy both approach zero. But suppose now that the variables x and y are themselves dependent on a parameter, t say. For example, we could be given $w = e^{xy}$

where $x = \cos t$ and $y = \sin t$, and wish to examine the behaviour of w as t varies. (In this case the point (x, y) would lie on a circle, so we would examine the behaviour of $w = e^{xy}$ for points on this circle.)

Suppose that δt is the increment in t that gives rise to the increments δx and δy . Then (dividing both sides of equation (2.3) by δt)

$$\frac{\delta w}{\delta t} \simeq \frac{\partial w}{\partial x} \frac{\delta x}{\delta t} + \frac{\partial w}{\partial y} \frac{\delta y}{\delta t},$$

and in the limit as δt approaches zero, the approximation becomes equality and we have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}.$$

This result is known as the **chain rule for functions of two variables**. (It is usually abbreviated to the **chain rule** when there is no danger of confusion.)

Suppose now that x and y are dependent on two variables, u and v say (rather than a single variable t). Then the previous argument applies with one minor difference. Keeping v fixed, the derivatives in the chain rule now become partial derivatives, and we have

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}.$$

Similarly, if we keep u fixed, then we have

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v}.$$

These are the relationships that enable us to change from one coordinate system to another. In Section 2.4, we demonstrate how to switch between derivatives involving Cartesian and polar coordinates.

Exercise 2.8

- Given $w = e^{xy}$, where $x = \sin t$ and $y = \cos t$, calculate $\frac{dw}{dt}$ first by substituting for x and y , and then by using the chain rule.
- Given a function $w = w(x, y)$, we make the substitutions $x = au + bv$ and $y = cu + dv$ (where a, b, c and d are constants) to express w as a function of u and v . Express the partial derivatives

$$\frac{\partial w}{\partial u}, \quad \frac{\partial w}{\partial v}, \quad \frac{\partial^2 w}{\partial u^2}, \quad \frac{\partial^2 w}{\partial u \partial v} \quad \text{and} \quad \frac{\partial^2 w}{\partial v^2}$$

in terms of partial derivatives with respect to x and y .

A pair of functions $(x(t), y(t))$ defines a parametric curve.

Note that we use partial derivatives (∂) when differentiating w with respect to x and y , but total derivatives (d) when differentiating x and y with respect to t . This is because $w(x, y)$ is a function of two variables, but $x(t)$ and $y(t)$ are functions of a single variable.

2.2.9 Multiple integrals

In Chapter 1, we introduced the idea of definite integration in a single variable as the limit of a sum. In this section we generalise this idea to integration over two or more variables. We begin with the case of integration over two variables.

Integrals over plane surfaces

We consider a function $f(\mathbf{r}) = f(x, y)$ defined on a region S of the Cartesian plane. We first divide the region into a large number, N , of small elements δA_m , as shown in Figure 2.7(a), so that the sum of these elements approximates the region S .

These small elements are not necessarily rectangular, although it is convenient to define them by a grid as in Figure 2.7.

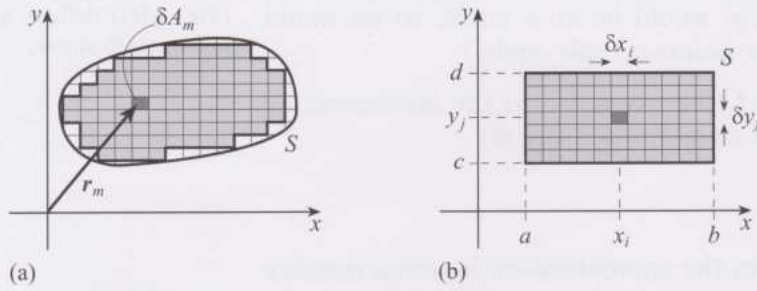


Figure 2.7 Regions divided into rectangular subregions

For each such region we choose a vector $\mathbf{r}_m$ corresponding to a point lying in δA_m . Then we define the integral of f over S to be the limit of the sum

$$\sum_{m=1}^N f(\mathbf{r}_m) \delta A_m$$

as N tends to infinity (and the size of the elements shrinks), so that

$$\int_S f(\mathbf{r}) dA = \lim_{N \rightarrow \infty} \sum_{m=1}^N f(\mathbf{r}_m) \delta A_m.$$

It is not essential to form the small regions using a Cartesian grid as in Figure 2.7, although this may enable us to write the integral in a more recognisable form.

The simplest case is when the region S is a rectangle, as shown in Figure 2.7(b). In this case we have $\delta A_m = \delta x_i \delta y_j$, and the sum over m can be found by first keeping j fixed and summing along a horizontal strip (i.e. over i), then summing over j . This process corresponds to first integrating $f(x, y)$ with respect to x while keeping y fixed, and then integrating the result with respect to y . Thus the integral becomes

$$\int_S f(\mathbf{r}) dA = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

(summing first across an arbitrary horizontal strip, and then vertically). Alternatively, we could reverse the order of summation to obtain the same value:

$$\int_S f(\mathbf{r}) dA = \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

(summing first up an arbitrary vertical strip, and then horizontally). These forms are known as **repeated integrals**.

In our first example we shall consider a particularly simple case, namely where $f(x, y) = 1$.

Example 2.11

Evaluate the repeated integral $\int_0^1 \left(\int_{x^2}^x dy \right) dx$.

Solution

We have

$$\begin{aligned} \int_0^1 \left(\int_{x^2}^x dy \right) dx &= \int_0^1 [y]_{x^2}^x dx = \int_0^1 (x - x^2) dx \\ &= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 = \frac{1}{6}. \quad \blacksquare \end{aligned}$$

Changing the order of summation *can* produce a different value, but only in very exceptional cases, which will not be encountered in this course.

Note that in later chapters we use the notation $\int dx f(x)$ rather than $\int f(x) dx$, because writing $\int dx \int dy f(x, y)$ in place of $\int \left(\int f(x, y) dy \right) dx$ avoids the need for brackets.

Example 2.12

Evaluate the integral $\int_S f(\mathbf{r}) dA$ where $f(\mathbf{r}) = f(x, y) = x^2 y$ and S is the rectangle with its sides parallel to the axes for which one vertex is at the origin and the diagonally opposite vertex is at the point $(1, 2)$, as shown in Figure 2.8.

Solution

We have

$$\begin{aligned} \int_S f(\mathbf{r}) dA &= \int_0^1 \left(\int_0^2 x^2 y dy \right) dx = \int_0^1 x^2 \left(\int_0^2 y dy \right) dx \\ &= \int_0^1 x^2 \left[\frac{1}{2} y^2 \right]_0^2 dx \\ &= 2 \int_0^1 x^2 dx = 2 \left[\frac{1}{3} x^3 \right]_0^1 = \frac{2}{3}. \quad \blacksquare \end{aligned}$$

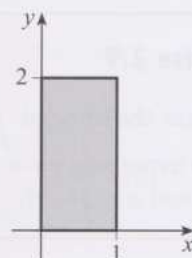


Figure 2.8

A slight adaptation of the method allows us to integrate over more complicated regions, as in the following example.

Example 2.13

Evaluate the integral $\int_S f(\mathbf{r}) dA$ where $f(\mathbf{r}) = f(x, y) = x^2 y$ and S is the region bounded by the curves $y = x^2$ and $y = 1 - x^2$, for $x \geq 0$, and the y -axis (so S is defined by the inequalities $y \geq x^2$, $y \leq 1 - x^2$ and $x \geq 0$; see Figure 2.9).

Solution

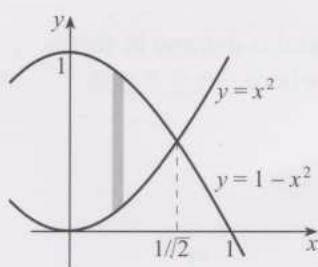


Figure 2.9 An elemental 'rectangular' area in the region S

Writing the integral as a repeated integral, where we integrate first along a strip from one curve to the other, as illustrated in Figure 2.9, we have

$$\begin{aligned} \int_S f(\mathbf{r}) dA &= \int_0^{1/\sqrt{2}} x^2 \left(\int_{x^2}^{1-x^2} y dy \right) dx \\ &= \int_0^{1/\sqrt{2}} x^2 \left[\frac{1}{2} y^2 \right]_{x^2}^{1-x^2} dx \\ &= \frac{1}{2} \int_0^{1/\sqrt{2}} x^2 ((1-x^2)^2 - x^4) dx \\ &= \frac{1}{2} \int_0^{1/\sqrt{2}} x^2 (1 - 2x^2) dx = \sqrt{2}/60. \quad \blacksquare \end{aligned}$$

It is often useful to draw a diagram of the integration region, as this helps in the construction of the integration limits.

You may wish to repeat this calculation with the order of integration reversed, i.e. perform the x integral first, and then the y integral. However, be warned that it will be necessary to alter the boundary conditions. It may be instructive to first examine the solution to Exercise 2.9, which solves a similar double integral in two ways.

Exercise 2.9

Evaluate the integral $\int_S x^2 dA$ where S is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$ and $x = 8$ (so $y \leq x$, $0 \leq x \leq 8$, $y \geq 0$ and $y \leq 16/x$).

Integrals involving polar coordinates

Recall that a point (x, y) in the plane can also be labelled by polar coordinates (r, θ) , where $x = r \cos \theta$ and $y = r \sin \theta$. See page 24 of Chapter 1.

The preceding method can be extended to integrals over areas defined in terms of polar coordinates.

Example 2.14

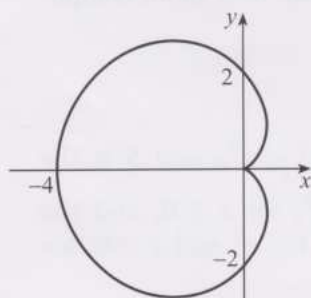


Figure 2.10 The cardioid $r = 2(1 - \cos \theta)$

The curve shown in Figure 2.10 is known as a *cardioid*, and is defined in terms of polar coordinates by the equation $r = 2(1 - \cos \theta)$ where $-\pi \leq \theta \leq \pi$.

Find the area of the region bounded by the curve.

Solution

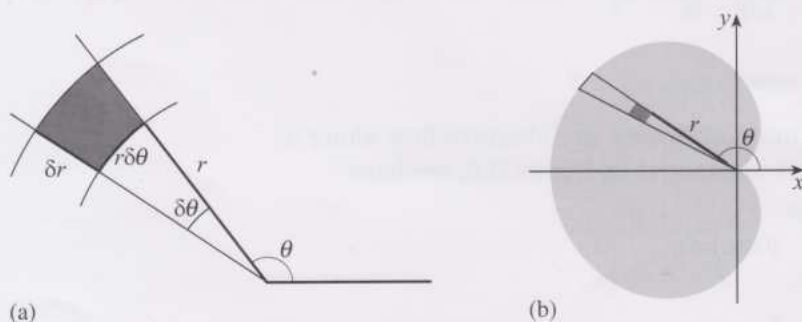


Figure 2.11 The region bounded by the cardioid divided into elements that are approximately rectangular

In order to find the area S bounded by the curve, we divide the region into small elements as shown in Figure 2.11(a). The element shown is approximately rectangular, so its area is approximately $r \delta \theta \delta r$. Summing over all

such elements produces an integral of the form $\iint_S r \, dr \, d\theta$, and writing this as a repeated integral we have

$$\begin{aligned}\iint_S r \, dr \, d\theta &= \int_{-\pi}^{\pi} \left(\int_0^{2(1-\cos \theta)} r \, dr \right) d\theta \\ &= 2 \int_0^{\pi} \left[\frac{1}{2} r^2 \right]_0^{2(1-\cos \theta)} d\theta \\ &= \int_0^{\pi} 4(1-\cos \theta)^2 d\theta = 6\pi. \quad \blacksquare\end{aligned}$$

The preceding example illustrates a general rule which allows us to transform between integrals of functions using Cartesian and polar coordinates. For any function f and integration region S ,

$$\iint_S f(x, y) \, dx \, dy = \iint_S r f(r \cos \theta, r \sin \theta) \, d\theta \, dr,$$

where the region S and the function f are expressed on the left-hand side in Cartesian coordinates, and on the right-hand side in polar coordinates. Transforming between coordinates is often essential for performing an integral, as the following example illustrates.

Example 2.15

Show that for $\alpha > 0$, the integral of the Gaussian function satisfies

$$I(\alpha) = \int_{-\infty}^{\infty} \exp(-\alpha x^2) \, dx = \sqrt{\frac{\pi}{\alpha}}.$$

This result was quoted in Chapter 1 as equation (1.4).

Solution

We first make the change of variable $u = \sqrt{\alpha}x$, so

$$I(\alpha) = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} \exp(-u^2) \, du = \frac{1}{\sqrt{\alpha}} I(1).$$

So it is necessary to evaluate $I(1)$. We do this by considering its square:

$$\begin{aligned}(I(1))^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} \, dy \right) \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \right) dy.\end{aligned}$$

We transform this integral to polar coordinates (r, θ) , remembering that $r^2 = x^2 + y^2$ and that the region of integration is the whole two-dimensional plane $0 \leq \theta \leq 2\pi$, $0 \leq r \leq \infty$:

$$\begin{aligned}(I(1))^2 &= \int_0^{\infty} \left(\int_0^{2\pi} e^{-r^2} r \, d\theta \right) dr = 2\pi \int_0^{\infty} e^{-r^2} r \, dr \\ &= -\pi \left[e^{-r^2} \right]_0^{\infty} = \pi.\end{aligned}$$

Hence we have shown that $I(1) = \sqrt{\pi}$, so $I(\alpha) = \sqrt{\frac{\pi}{\alpha}}$, as required. $\blacksquare$

Exercise 2.10

Evaluate the integral $\int_S r^2 \, dr \, d\theta$, where S is the region defined by $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$.

Triple integrals and multiple integrals in general

The notion of a double integral can be generalised easily to integrals of functions of three variables. We consider a function $f(\mathbf{r}) = f(x, y, z)$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is defined on a region R of three-dimensional space. We then divide the region R into a large number, N say, of small regions, each of volume ΔV_i , and the triple integral is defined as a limit of a sum:

$$\iiint_R f(x, y, z) dV = \lim_{N \rightarrow \infty} \sum_{i=1}^N f(x_i, y_i, z_i) \Delta V_i.$$

Such integrals are often known as volume integrals.

In this course multiple integrals generally arise from theoretical considerations, and we rarely need to evaluate them except in cases where the symmetry, or other factors, makes the calculation particularly easy. However, as an aid to understanding, it may help to see how we might undertake such a calculation. For this purpose we shall choose a very simple region.

Example 2.16

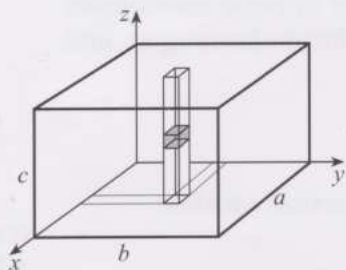


Figure 2.12 The rectangular box R divided into elemental subvolumes

In this example we suppose that R is the rectangular box defined by the inequalities

$$0 \leq x \leq a, \quad 0 \leq y \leq b, \quad 0 \leq z \leq c$$

(as shown in Figure 2.12), and that

$$f(x, y, z) = (x^2 + y^2)e^{-z}.$$

We then have

$$\begin{aligned} \iiint_R f(x, y, z) dV &= \int_0^a \left(\int_0^b \left(\int_0^c (x^2 + y^2)e^{-z} dz \right) dy \right) dx \\ &= \int_0^a \left(\int_0^b (x^2 + y^2)(1 - e^{-c}) dy \right) dx \\ &= (1 - e^{-c}) \int_0^a \left(\int_0^b (x^2 + y^2) dy \right) dx \\ &= (1 - e^{-c}) \int_0^a \left[x^2 y + \frac{1}{3} y^3 \right]_0^b dx \\ &= (1 - e^{-c}) \int_0^a \left(bx^2 + \frac{1}{3} b^3 \right) dx \\ &= \frac{1}{3} (1 - e^{-c}) ab(a^2 + b^2). \quad \blacksquare \end{aligned}$$

Integrals over surfaces that are not planar

We define integrals over surfaces that are not planar in a similar fashion, first dividing the surface into a large number of small elements, then forming a sum, and finally taking the limit. In practice we try to choose a coordinate system which allows us to write the integral as a repeated integral, as in the following example.

We assume that you have met vectors before; we shall refresh your memory in the next section.

Example 2.17

Given $f(\mathbf{r}) = 3z - \sqrt{x^2 + y^2}$, evaluate the integral $\int_S f(\mathbf{r}) dA$ where S is the curved surface of a cylinder of height $h/2$ and radius R with its axis on the z -axis and its base on the plane $z = h/2$.

Solution

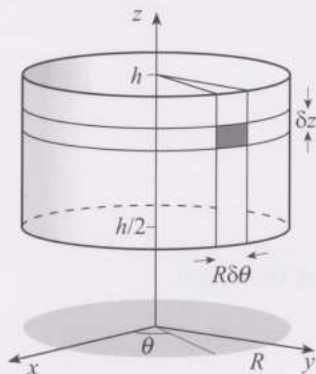


Figure 2.13 The curved surface of a cylinder divided into elemental rectangles

In this case it is helpful to choose cylindrical polar coordinates (r, θ, z) where $x = r \cos \theta$ and $y = r \sin \theta$. Then $f(\mathbf{r}) = 3z - \sqrt{x^2 + y^2} = 3z - r$, and an area element on the surface is $\delta A = R \delta \theta \delta z$ (see Figure 2.13).

On the curved surface of the cylinder we have $r = R$, so

$$\begin{aligned} \int_S f(\mathbf{r}) dA &= \int_{h/2}^h \left(\int_0^{2\pi} (3z - R) R d\theta \right) dz \\ &= \int_{h/2}^h \left(\int_0^{2\pi} d\theta \right) (3zR - R^2) dz \\ &= 2\pi \int_{h/2}^h (3zR - R^2) dz \\ &= \frac{1}{4} h \pi (9h - 4R) R. \quad \blacksquare \end{aligned}$$

Most of the integrals of this kind that will arise in this course are particularly simple. For example, if $f(\mathbf{r}) = 1$ and S is a sphere of radius ρ with its centre at the origin, then

$$\int_S f(\mathbf{r}) dA = \int_S dA = 4\pi\rho^2.$$

Similarly, if $f(\mathbf{r}) = |\mathbf{r}|^3 + 2$, then

$$\int_S f(\mathbf{r}) dA = (\rho^3 + 2) \int_S dA = 4\pi\rho^2(\rho^3 + 2).$$

Recall that a sphere of radius ρ has surface area $4\pi\rho^2$.

End-of-section Exercises

Exercise 2.11

Find the partial derivatives $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial y^2}$ if $w = \frac{1}{\sqrt{x^2 + y^2}}$.

Exercise 2.12

Verify the mixed derivative theorem (see page 66) for the function

$$F(x, y) = \frac{x}{x^2 + y^2}.$$

Exercise 2.13

Show that each of the following functions is a solution of

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,$$

the two-dimensional form of Laplace's equation in Cartesian coordinates.

(a) $w = e^x \cos y$ (b) $w = x^3 - 3xy^2$

This equation will be discussed briefly in Section 2.4.

Exercise 2.14

Find and classify the stationary points of the function

$$F(x, y) = 1 - 22x - 12y + 10x^2 + 2xy + 5y^2.$$

Exercise 2.15

Find the point on the plane $2x - y + 2z = 16$ that is nearest the origin.

Exercise 2.16

Given $w = \sqrt{x^2 + y^2 + z^2}$, calculate

$$\frac{\partial^3 w}{\partial x \partial y \partial z}, \quad \frac{\partial^3 w}{\partial z \partial x \partial y} \quad \text{and} \quad \frac{\partial^3 w}{\partial y \partial z \partial x}.$$

Exercise 2.17

Evaluate the repeated integral $\int_0^\pi \left(\int_0^{\cos \theta} \rho \sin \theta \, d\rho \right) d\theta$.

Exercise 2.18

The region S in the (x, y) -plane is bounded by the three lines $x = 3$, $y = x/3$ and $y = 0$. Write the integral of the function $\exp(x^2)$ over the region S as a repeated integral in two ways, namely with the x integral performed first, and then with the y integral performed first. Use one of these to evaluate $\iint_S \exp(x^2) \, dA$.

This exercise is optional.

2.3 Vector calculus

In this section we shall be concerned mainly with the behaviour of functions whose domains are sets of vectors, so we now revise some elementary properties of vectors.

2.3.1 Vector algebra

We assume that you are familiar with the fundamental notions of vector algebra. This is essentially the material that you are likely to have met in previous Open University courses, but for students who have gained their knowledge of the subject from a different source, we provide a very brief summary and explain our notation.

A brief summary

Scalars are quantities that are defined by their magnitude, such as mass and length, and we use the term to distinguish them from **vector** quantities, which have both magnitude and direction, such as velocity. Vectors are often

denoted in printed material by bold type, for example $\mathbf{v}$ or $\mathbf{F}$, or by underlining in handwritten material, $\underline{v}$ or $\underline{F}$ (although several other conventions are in common use). They are often specified by pairs of numbers in two dimensions, and triples of numbers in three dimensions: for example, (x, y) in two-dimensional Cartesian coordinates, or (x, y, z) in three dimensions.

The following statements are made in the context of three-dimensional vectors, but similar results apply in two dimensions. The vector (x, y) in two dimensions is often equated with the vector $(x, y, 0)$ in three dimensions.

A vector $\mathbf{r} = (x, y, z)$ may be multiplied by a scalar α to give

$$\alpha \mathbf{r} = (\alpha x, \alpha y, \alpha z),$$

and two vectors in Cartesian form may be added to give

$$\mathbf{r}_1 + \mathbf{r}_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

The **zero vector** $\mathbf{0}$ corresponds to $(0, 0, 0)$ in Cartesian coordinates.

The **magnitude** or **length** of a vector $\mathbf{r} = (x, y, z)$ is denoted by $|\mathbf{r}|$ and defined to be $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. A **unit vector** is a vector of magnitude 1. The unit vectors in the directions of the Cartesian coordinate axes are labelled $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, so a three-dimensional Cartesian vector $\mathbf{r} = (x, y, z)$ may also be written in the form $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The scalars x , y and z are known as the **Cartesian components** of the vector $\mathbf{r}$.

We write a 'hat' symbol over a vector to denote the fact that it is a unit vector. Thus a unit vector in the direction of a given vector $\mathbf{r}$ is denoted by $\hat{\mathbf{r}}$, where $\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$.

The scalar product

The scalar (or dot) product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta, \quad (2.4)$$

where θ is the angle between the vectors. Using a unit vector $\hat{\mathbf{b}}$, the scalar product $\mathbf{a} \cdot \hat{\mathbf{b}}$ represents the length L shown in Figure 2.14; this is known as the **projection** of the vector $\mathbf{a}$ in the direction of $\hat{\mathbf{b}}$. Note that the scalar product of two vectors is a scalar.

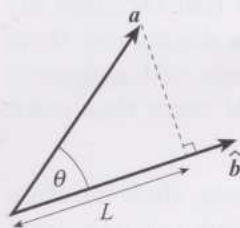


Figure 2.14 The projection of a vector $\mathbf{a}$ in the direction of $\hat{\mathbf{b}}$

In Cartesian components, the scalar product of vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ can be written as

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3. \quad (2.5)$$

Note that it is not immediately obvious that equations (2.4) and (2.5) are equivalent. However, you should be prepared to use either definition as the need arises.

The following list includes some useful properties of the scalar product.

Strictly speaking, (x, y, z) is a *representation* of the vector $\mathbf{r}$ rather than the vector itself, but we shall not be concerned with such fine distinctions.

Some properties of vectors

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \quad \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0,$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

$$= (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \cdot (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k})$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= \mathbf{b} \cdot \mathbf{a},$$

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\mathbf{a} \cdot \mathbf{a}},$$

$$|\lambda \mathbf{a}| = |\lambda| |\mathbf{a}| \text{ for any scalar } \lambda,$$

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c},$$

$$(\lambda + \mu) \mathbf{a} = \lambda \mathbf{a} + \mu \mathbf{a} \text{ for any scalars } \lambda \text{ and } \mu.$$

Example 2.18

For a fixed vector $\mathbf{v}$, and any unit vector $\hat{\mathbf{n}}$, show that $\mathbf{v} \cdot \hat{\mathbf{n}}$ has maximum value $|\mathbf{v}|$, when $\hat{\mathbf{n}}$ is chosen to be parallel to $\mathbf{v}$.

Solution

Using equation (2.4),

$$\mathbf{v} \cdot \hat{\mathbf{n}} = |\mathbf{v}| |\hat{\mathbf{n}}| \cos \theta = |\mathbf{v}| \cos \theta.$$

For any given $\mathbf{v}$, this has a maximum value $|\mathbf{v}|$ when $\cos \theta = 1$. This corresponds to $\theta = 0$, i.e. to choosing $\hat{\mathbf{n}}$ to be parallel to $\mathbf{v}$. ■

Exercise 2.19

- (a) Find the cosine of the angle θ between the vectors $\mathbf{a} = (1, 2, -1)$ and $\mathbf{b} = (2, 1, 1)$.
- (b) Find the magnitude of the vector $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$.

2.3.2 Scalar fields

In applied mathematics, real functions of two and three real variables are often known as scalar fields. The term ‘scalar’ is used to distinguish them from the ‘vector fields’ that we shall discuss shortly. We begin with a physical situation which may help you to understand the general cases that come later.

If a drop of ink falls onto a large sheet of wet blotting paper, then the stain will spread. The density F of the stain (i.e. the mass of ink per unit area) at a particular instant of time might be represented quite well by a function

$$F(x, y) = A \exp[-\alpha(x^2 + y^2)],$$

where A and α are positive constants (and the blotting paper is assumed to lie in the (x, y) -plane with the centre of the stain at the origin). In this case $F(x, y)$ is an example of a two-dimensional scalar field.

Suppose that time is kept fixed, and imagine that we follow a curve drawn on the paper through the ink stain: we shall notice a change in the ink density as we traverse the path. At any particular point on the path, we intend to determine the direction in which the ink density increases most rapidly.

If we represent $F(x, y)$ as a surface, as in Figure 2.15(b), then the height of the surface above the point (x, y) represents the density of the stain at that point.

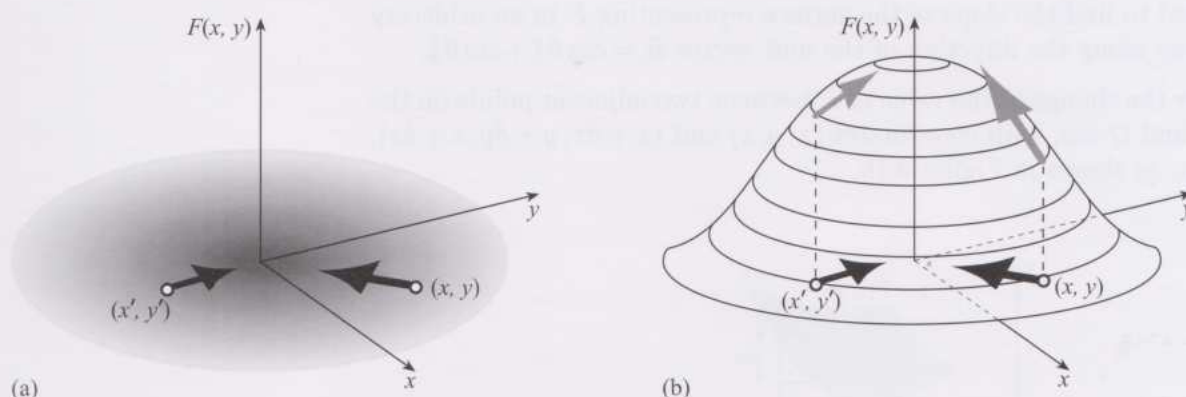


Figure 2.15 (a) The ink stain in the (x, y) -plane; (b) the surface representing $F(x, y) = A \exp[-\alpha(x^2 + y^2)]$

The gradient

At most points on the surface representing F we should be able to determine both the direction in which the slope is greatest and the magnitude of this greatest slope. We can then associate a vector having this magnitude and direction with each point (x, y) . Since the magnitude and direction of greatest slope will vary from point to point, the magnitude and direction of the vector representing this will also vary from point to point. In other words, this vector will be a function of position (x, y) . This vector function is known as the **gradient** of F , denoted by ∇F ; we sometimes write $\nabla F(x, y)$ to emphasise its positional dependence.

For our ink stain illustration, ∇F always points in the direction for which the ink density increases most rapidly. In Figure 2.15 we show ∇F at two points, (x, y) and (x', y') , by the darker arrows. If we were to draw the level curves for the surface representing F , then we would obtain a diagram very similar to Figure 2.2 (page 59), and we would see that the gradient at each point is perpendicular to the tangent to the level curve that passes through that point.

We now formally define the gradient of a function, and state some properties associated with it.

Read ∇ as 'grad'.
Some authors write **grad** F instead of ∇F .

The lighter arrows point up the lines of greatest slope, and the darker arrows are their projections onto the (x, y) -plane.

The gradient

The gradient of any function $F(x, y)$ is defined as

$$\nabla F = \frac{\partial F}{\partial x} \mathbf{i} + \frac{\partial F}{\partial y} \mathbf{j}.$$

At each point (x, y) , ∇F is a vector which points in the direction of maximum slope (uphill) of F ; its magnitude $|\nabla F|$ is the value of the maximum slope at that point.

Additionally, if $\hat{n}$ is any unit vector, then $\nabla F \cdot \hat{n}$ gives the slope of F in the direction of $\hat{n}$.

The proof of these properties is presented as an optional example.

Example 2.19*This example is optional.*

Prove the aforementioned properties of ∇F .

Solution

We first need to find the slope of the surface representing F in an arbitrary direction, say along the direction of the unit vector $\hat{n} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$.

We consider the change in the value of F between two adjacent points on the surface, P and Q say, with coordinates (x, y, z) and $(x + \delta x, y + \delta y, z + \delta z)$, respectively, as shown in Figure 2.16.

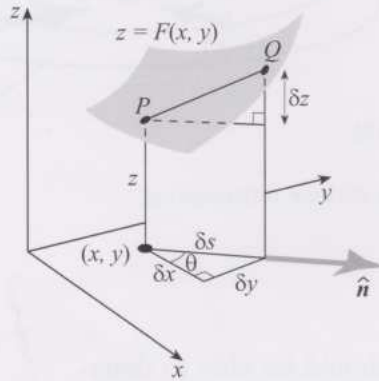


Figure 2.16 The points P and Q on the surface representing F

We now keep P fixed and consider the slope of the line PQ . If we let $\delta s = \sqrt{\delta x^2 + \delta y^2}$, then we see from Figure 2.16 that the slope of the line joining P and Q is given by $\frac{\delta z}{\delta s}$. We know (from equation (2.1)) that

$$\delta z \simeq F_x \delta x + F_y \delta y,$$

where the partial derivatives are evaluated at P . Therefore

$$\text{slope of } PQ = \frac{\delta z}{\delta s} \simeq F_x \frac{\delta x}{\delta s} + F_y \frac{\delta y}{\delta s} = F_x \cos \theta + F_y \sin \theta.$$

This result holds in the limit as $\delta s \rightarrow 0$, so

$$\text{slope in the direction of } \hat{n} = (F_x \mathbf{i} + F_y \mathbf{j}) \cdot \hat{n} = \nabla F \cdot \hat{n}.$$

So, keeping P fixed so that F_x and F_y are constant, we can calculate the slope in any direction simply by varying the direction of $\hat{n}$. From Example 2.18, we see that the maximum slope has value $|\nabla F|$, and this occurs when ∇F is parallel to $\hat{n}$. ■

Of course, this argument fails if $F_x = F_y = 0$, since then there is no direction of maximum slope.

We mentioned earlier that at every point ∇F is perpendicular to the tangents of the level curves of F , but we shall not prove this here.

In the case of the ink stain example, we have $F(x, y) = A \exp[-\alpha(x^2 + y^2)]$, so

$$\frac{\partial F}{\partial x} = -2\alpha A x \exp[-\alpha(x^2 + y^2)]$$

and

$$\frac{\partial F}{\partial y} = -2\alpha A y \exp[-\alpha(x^2 + y^2)],$$

therefore

$$\nabla F = -2\alpha A \exp[-\alpha(x^2 + y^2)] (x\mathbf{i} + y\mathbf{j}).$$

This is a vector in the same direction as a vector from the point (x, y) to the origin, which is what we would expect from the symmetry shown in Figure 2.15.

Similar ideas can also be applied to a three-dimensional scalar field $F(x, y, z)$, for which we define

$$\nabla F = \frac{\partial F}{\partial x} \mathbf{i} + \frac{\partial F}{\partial y} \mathbf{j} + \frac{\partial F}{\partial z} \mathbf{k}.$$

It is sometimes useful to regard $\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$ as an operator that can be applied to functions, in which case it is known as a **vector differential operator**.

∇ is known as **del** or **nabla** in some texts.

Example 2.20

Calculate ∇F for the function $F(x, y, z) = x^2 y \sin z$.

Solution

We have

$$\frac{\partial F}{\partial x} = 2xy \sin z, \quad \frac{\partial F}{\partial y} = x^2 \sin z \quad \text{and} \quad \frac{\partial F}{\partial z} = x^2 y \cos z,$$

so

$$\begin{aligned} \nabla F &= \frac{\partial F}{\partial x} \mathbf{i} + \frac{\partial F}{\partial y} \mathbf{j} + \frac{\partial F}{\partial z} \mathbf{k} \\ &= (2xy \sin z) \mathbf{i} + (x^2 \sin z) \mathbf{j} + (x^2 y \cos z) \mathbf{k}. \quad \blacksquare \end{aligned}$$

Example 2.21

Calculate the gradient of the function $F(x, y) = x^2 - xy + 3y^2$ at the point $(2, -1)$.

Solution

The point $(2, -1)$ refers to a point in the domain of F (i.e. the point $(2, -1, 9)$ on the corresponding surface). $F_x(x, y) = 2x - y$ and $F_y(x, y) = -x + 6y$, so $F_x(2, -1) = 5$ and $F_y(2, -1) = -8$. It follows that $\nabla F(2, -1) = 5\mathbf{i} - 8\mathbf{j}$.

A schematic representation of ∇F , together with the level curves of F , is shown in Figure 2.17. The arrow at each point represents the vector ∇F at that point. Note that at each point, ∇F is perpendicular to the tangent to the level curve of F .

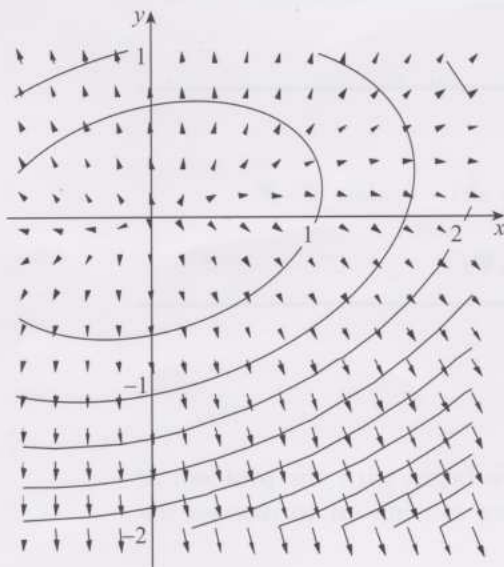


Figure 2.17 The level curves and gradient of $F(x, y) = x^2 - xy + 3y^2$ ■

Exercise 2.20

Calculate the gradient of the function $F(x, y) = (x^2 + 2y^2) \sin[(x + y)\pi]$ at the point $(1, 1)$.

2.3.3 Vector fields

A vector-valued function $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ in two dimensions, or $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$ in three dimensions, is often known as a **vector field**, and we commonly make abbreviations like $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ when the choice of independent variables is clear.

We saw in the previous subsection that the gradient of a function is a vector field; but not all vector fields arise as gradients. Physical examples of vector fields include fluid velocity (where the vector at each point is equal in magnitude and direction to the fluid velocity), and the force on a charged particle in an electric field (where the vector at each point is equal to the force on the particle at that point).

The spatial variation of a vector field is inevitably more complicated than that of a scalar field: for example, in three dimensions, it involves the nine partial derivatives

$$\frac{\partial F_1}{\partial x}, \frac{\partial F_1}{\partial y}, \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x}, \frac{\partial F_2}{\partial y}, \frac{\partial F_2}{\partial z}, \frac{\partial F_3}{\partial x}, \frac{\partial F_3}{\partial y}, \frac{\partial F_3}{\partial z}.$$

These partial derivatives completely describe the spatial variation of $\mathbf{F}$, and certain combinations of them occur quite often. The combination that interests us here corresponds to the scalar field known as the **divergence** of $\mathbf{F}$, defined as the scalar product of the operator ∇ with the vector field $\mathbf{F}$:

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \cdot (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \end{aligned}$$

Some authors use the notation $\operatorname{div} \mathbf{F}$ in place of $\nabla \cdot \mathbf{F}$.

Note that although $\mathbf{F}$ is a vector field, $\nabla \cdot \mathbf{F}$ is a scalar field. The importance of $\nabla \cdot \mathbf{F}$ will become clearer when we discuss the Gauss divergence theorem; however, this theorem involves integration over surfaces and volumes, so first we need to say a little more about integration.

Exercise 2.21

- (a) Given $\mathbf{F}(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{1/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, calculate $\nabla \cdot \mathbf{F}$.
- (b) Given $f(x, y, z) = x^2 + y^2 + z^2$, calculate $\nabla \cdot (\nabla f)$.

2.3.4 Integration

We now return to the topic of integration. We begin with the problem of how to determine the length of a curve, before moving on to discuss the integration of vector-valued functions.

The length of a curve

We continue with the idea of integration being the limit of a sum.

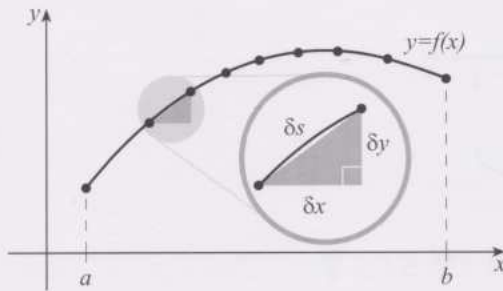


Figure 2.18 A curve divided into short subintervals

Suppose that we wish to determine the length L of the curve $y = f(x)$ between $x = a$ and $x = b$, as shown in Figure 2.18. One approach might be to divide the curve into a large number (say N) of small segments that are almost linear. The sum of the approximate lengths of these small segments would then give us an approximation to the total length of the curve. If we increase the number of such subdivisions, we might expect to obtain an increasingly accurate approximation to the total length. In other words, the length of the curve could be represented as the limit of a sum, i.e. as an integral.

For the length δs_k of the k th segment, we have $\delta s_k^2 \simeq \delta x_k^2 + \delta y_k^2$. So

$$L \simeq \sum_{k=1}^N \delta s_k = \sum_{k=1}^N \sqrt{\delta x_k^2 + \delta y_k^2} = \sum_{k=1}^N \delta x_k \sqrt{1 + \left(\frac{\delta y_k}{\delta x_k}\right)^2},$$

and in the limit as $N \rightarrow \infty$ we have

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Example 2.22

Calculate the length of the graph of the function $y = x^{3/2}$ from the origin to the point $(1, 1)$.

Solution

The required length is

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2} dx \\ &= \frac{1}{2} \int_0^1 \sqrt{4 + 9x} dx \\ &= \frac{1}{27} \left[(4 + 9x)^{3/2} \right]_0^1 \\ &= \frac{1}{27} (13\sqrt{13} - 8). \quad \blacksquare \end{aligned}$$

The same idea can be extended to a curve that is not the graph of a function (i.e. not of the form $y = f(x)$), for example the curve shown in Figure 2.19. Such curves often arise in the parametrised form $x = x(t)$, $y = y(t)$ with $t_1 \leq t \leq t_2$, and are called **parametric curves**. A *smooth curve* is defined to be a curve for which the functions $x(t)$ and $y(t)$ have continuous derivatives.

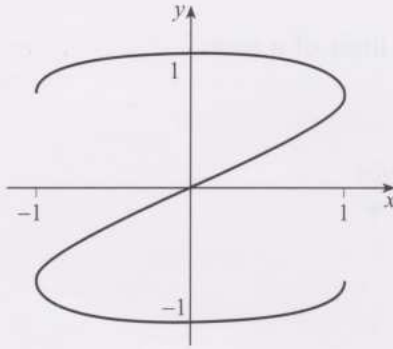


Figure 2.19 The curve $x(t) = \sin 2t$, $y(t) = \sin t$ with $-\frac{3\pi}{4} \leq t \leq \frac{3\pi}{4}$

The curve in Figure 2.19 can be specified in parametrised form by putting $x(t) = \sin 2t$ and $y(t) = \sin t$ with $-\frac{3\pi}{4} \leq t \leq \frac{3\pi}{4}$.

In general, if s denotes the distance along the curve from any fixed point, we have $\delta s^2 \simeq \delta x^2 + \delta y^2$ as before, therefore

$$\left(\frac{\delta s}{\delta t}\right)^2 \simeq \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2,$$

and in the limit as $\delta t \rightarrow 0$ we have

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

Thus $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$, so for the length L of the curve shown in Figure 2.19 we have

$$\begin{aligned} L &= \int_{-3\pi/4}^{3\pi/4} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_{-3\pi/4}^{3\pi/4} \sqrt{4\cos^2 2t + \cos^2 t} dt \simeq 6.91784. \end{aligned}$$

The integral was evaluated by numerical methods.

Such **line integrals** arise in a number of areas of applied mathematics. For example, it is relatively easy to calculate the work done when an object is pushed up a straight path against a constant force $\mathbf{F}$, as shown in Figure 2.20. We simply multiply the length of the path d by the component F of the force along the line of the path.

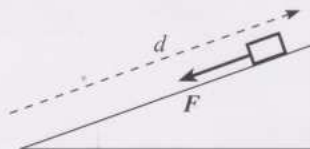


Figure 2.20 A constant force acting along a straight path

Imagine now that we push an object up a winding cliff path, and that we wish to find the total work done. In theory we could divide the path into a large number of almost linear sections, then calculate the work done when the object is pushed along the many short straight paths, and thus obtain an approximation to the total work done. We expect the approximation to improve as the number of subdivisions increases.

In the following subsection we shall make this idea a little more precise.

Line integrals of vector fields

Suppose that we are given a parametric curve C defined by a vector $\mathbf{r}(t)$, so the points on the path are determined by the values of the parameter t . Now suppose that a particle is propelled along the curve by a force $\mathbf{F}(\mathbf{r})$, and that we wish to calculate the work done by the force. If a constant force $\mathbf{F}$ acts on a particle and displaces it through a vector distance $\delta\mathbf{s}$, then the work done by the force is $\mathbf{F} \cdot \delta\mathbf{s}$. So in order to calculate the total work done by the variable force $\mathbf{F}(\mathbf{r})$ as it moves along a curve, we approximate the path by a large number of straight-line subpaths on which the force and the direction are approximately constant (see Figure 2.21). We then have

$$\text{total work done} \simeq \sum_{k=1}^N \mathbf{F}(\mathbf{r}_k) \cdot \delta\mathbf{r}_k.$$

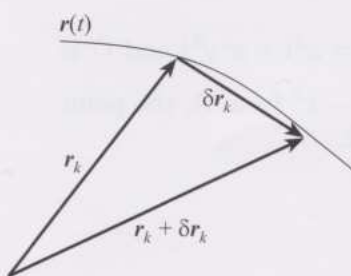


Figure 2.21 A subpath $\delta\mathbf{r}_k$ on the curve $\mathbf{r}(t)$

This approximation will improve as N increases (and the maximum length of the subpaths decreases), leading us to define a *line integral* as

$$I = \lim_{N \rightarrow \infty} \sum_{n=1}^N \mathbf{F}(\mathbf{r}_n) \cdot \delta\mathbf{r}_n,$$

which we shall write in a more recognisable form shortly.

Such integrals can also be used to represent quantities other than the work done.

Suppose now that C is a curve in three dimensions, and that it can be parametrised by writing

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

with the beginning and end points of C corresponding to $t = a$ and $t = b$, respectively. On C we have $\delta\mathbf{r} = \delta x\mathbf{i} + \delta y\mathbf{j} + \delta z\mathbf{k}$, so $\frac{\delta\mathbf{r}}{\delta t} = \frac{\delta x}{\delta t}\mathbf{i} + \frac{\delta y}{\delta t}\mathbf{j} + \frac{\delta z}{\delta t}\mathbf{k}$, where δt is the increment in t that gives rise to the increment $\delta\mathbf{r}$, therefore

$$\delta\mathbf{r} \simeq \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \right) \delta t = (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}) \delta t.$$

If $\mathbf{F}(\mathbf{r}) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$, so that on C the components of $\mathbf{F}$ are functions of t , then

$$\begin{aligned} & \sum_{n=1}^N \mathbf{F}(\mathbf{r}_n) \cdot \delta\mathbf{r}_n \\ & \simeq \sum_{n=1}^N (F_1(t_n)\mathbf{i} + F_2(t_n)\mathbf{j} + F_3(t_n)\mathbf{k}) \cdot (x'(t_n)\mathbf{i} + y'(t_n)\mathbf{j} + z'(t_n)\mathbf{k}) \delta t_n, \end{aligned}$$

where $t_1 = a$ and $t_N = b$, so $N\delta t = b - a$.

In two dimensions,
 $\mathbf{r}(t) = (x(t), y(t))$;
 in three dimensions,
 $\mathbf{r}(t) = (x(t), y(t), z(t))$.

The $\mathbf{i}$ component of $\mathbf{F}$ is $F_1(x(t), y(t), z(t)) = F_1(t)$, and similarly for the other components.

In the limit as $N \rightarrow \infty$,

$$I = \int_a^b \left(F_1(t) \frac{dx}{dt} + F_2(t) \frac{dy}{dt} + F_3(t) \frac{dz}{dt} \right) dt = \int_a^b \mathbf{F} \cdot \left(\frac{d\mathbf{r}}{dt} \right) dt$$

(which is just a 'normal' integral with respect to t).

The line integral I is usually written as

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r},$$

where it is understood that the curve C has been parametrised so that $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a function of a parameter t .

The following example (in two dimensions) may help to clarify the notion of a line integral.

Example 2.23

Evaluate the line integral $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ where $\mathbf{F}(\mathbf{r}) = x^2\mathbf{i} + x^2y^2\mathbf{j}$ and C is the part of a parabola defined by the equation $y = 1 - x^2$ from A , the point $(0, 1)$, to B , the point $(1, 0)$, as shown in Figure 2.22.

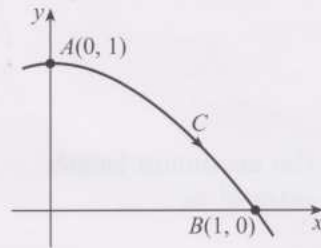


Figure 2.22 Part of the parabola $y = 1 - x^2$

Solution

First we parametrise the curve by writing $x(t) = t$ and $y(t) = 1 - t^2$, with $0 \leq t \leq 1$. Then $t = 0$ and $t = 1$ correspond to the points A and B , respectively. Also notice that $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -2t$. We then have

$$\mathbf{F}(\mathbf{r}) = x^2\mathbf{i} + x^2y^2\mathbf{j} = t^2\mathbf{i} + t^2(1 - t^2)^2\mathbf{j}$$

and

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt} (x(t)\mathbf{i} + y(t)\mathbf{j}) = \mathbf{i} - 2t\mathbf{j},$$

so

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_0^1 (t^2\mathbf{i} + t^2(1 - t^2)^2\mathbf{j}) \cdot (\mathbf{i} - 2t\mathbf{j}) dt \\ &= \int_0^1 (t^2 - 2t^3(1 - t^2)^2) dt = \frac{1}{4}. \quad \blacksquare \end{aligned}$$

The evaluation of such integrals in three dimensions is very similar.

Exercise 2.22

- (a) Calculate the length of the curve $x = t^2$, $y = t^3$ from $t = 0$ to $t = 4$.

- (b) At time t seconds, the position of a particle is given by $x = t$, $y = t^2/\sqrt{2}$, $z = t^3/3$. How far does the particle travel in the two seconds from $t = 0$ to $t = 2$?
- (c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ when $\mathbf{F}(x, y, z) = x^2y\mathbf{i} + z\mathbf{j} + xy^2\mathbf{k}$ and C is the straight line joining the origin to the point $(1, 2, 3)$.

2.3.5 The Gauss divergence theorem

We can now move on to discuss the Gauss divergence theorem, which involves an integral over a volume and an integral over the corresponding surface.

The flux of a vector field across a surface

The concept of the flux of a vector field across a surface arises in many areas of applied mathematics, but perhaps the easiest to visualise is the flow of a fluid or gas across a surface. Imagine the flow of a fluid in three dimensions, where at each point P with position vector $\mathbf{r} = (x, y, z)$ the fluid has velocity $\mathbf{F}(\mathbf{r})$; i.e. the fluid velocity is described by the vector field $\mathbf{F}(\mathbf{r})$. Now imagine that we immerse a wire mesh in the fluid, and that we wish to estimate the volume of fluid passing through the mesh in one second. The situation is depicted in Figure 2.23(a), where the mesh is shaped rather like the visor of a motorcycle helmet.

This is not a real wire mesh, but rather a mathematical construct with infinitesimally thin wires, and ultimately with an infinitely fine mesh.

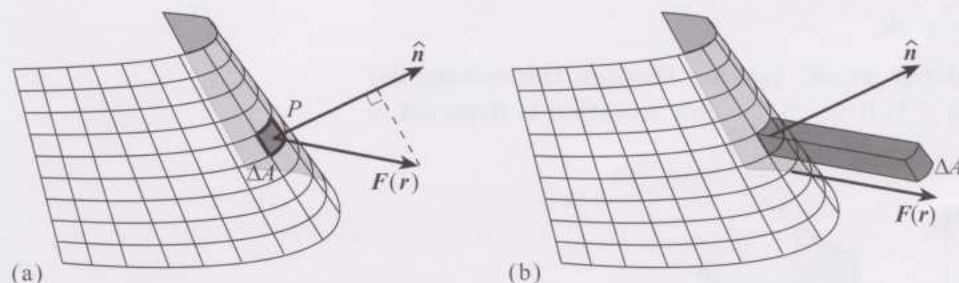


Figure 2.23 A surface approximated by a 'wire mesh', and the volume of fluid passing through an aperture in one second

First we consider just one small aperture in the mesh, and calculate the volume of fluid passing through it. Suppose that the aperture is at position P with area ΔA , and that fluid passing through this aperture has velocity $\mathbf{F}$. Also suppose that the vector $\hat{\mathbf{n}}$ is a unit normal to the surface at P .

A vector perpendicular to the tangent plane at a point P on a surface is known as the **normal to the surface** at P . At every point on a smooth surface there are two normals, one 'into the surface' and one 'out of the surface'. If the former is denoted by $\hat{\mathbf{n}}$, then the latter will be $-\hat{\mathbf{n}}$. For a closed surface (i.e. a surface without a boundary) we usually take the outward normal to be positive.

If the aperture is small enough, then the fluid velocity will not vary across it.

In general, the fluid velocity and the unit normal will not be parallel, and in Figure 2.23(b) the shaded volume represents the fluid passing through the area ΔA in one second. This is a parallelepiped with base area ΔA and length F (see Figure 2.24).

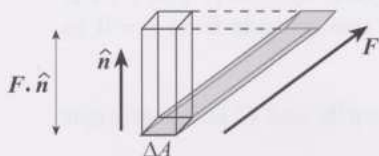


Figure 2.24

From the diagram we see that the volume of this parallelepiped is equal to the volume of a box with base area ΔA and height $\mathbf{F} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit normal at P . Thus it is the component of velocity perpendicular to the mesh that is relevant, and the volume of fluid passing through the aperture in one second is approximately $\mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{n}} \Delta A$.

If we sum over all apertures, and form the limit of the sum as the size of the mesh shrinks to zero and the number of subdivisions increases, we obtain an integral

$$\int_S \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{n}} dA.$$

This integral is the **flux** of the vector field $\mathbf{F}$ across the surface S , which in this case represents the volume of liquid crossing the surface in one second.

It is often convenient to abbreviate $\hat{\mathbf{n}} dA$ to $d\mathbf{A}$, in which case the integral is written $\int_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{A}$.

The vector $d\mathbf{A} = \hat{\mathbf{n}} dA$ has magnitude dA and direction normal to the surface. Since surfaces are in general curved, the direction of $\hat{\mathbf{n}}$ and $d\mathbf{A}$ will depend on $\mathbf{r}$.

Example 2.24

If the velocity of a fluid is given by the vector field

$$\mathbf{F} = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k},$$

calculate the volume of fluid per second passing through the rectangular area S given by $x = 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$. This situation is depicted in Figure 2.25.

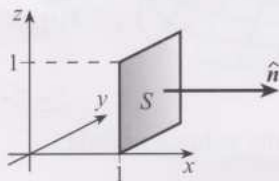


Figure 2.25

Solution

The unit normal to S is given by $\hat{\mathbf{n}} = \mathbf{i}$, so $d\mathbf{A} = \mathbf{i} dA = \mathbf{i} dy dz$. Hence the volume of fluid passing through S is

$$\begin{aligned} V &= \int_S \mathbf{F}(1, y, z) \cdot d\mathbf{A} \\ &= \int_0^1 \left(\int_0^1 (y^2z^2\mathbf{i} + yz^2\mathbf{j} + y^2z\mathbf{k}) \cdot \mathbf{i} dy \right) dz \\ &= \int_0^1 \left(\int_0^1 y^2z^2 dy \right) dz = \int_0^1 \frac{1}{3}z^2 dz = \frac{1}{9}. \end{aligned}$$

Note that we could have chosen $\hat{\mathbf{n}} = -\mathbf{i}$, in which case we would be calculating the volume of fluid passing through S in the $-\mathbf{i}$ direction, which would simply give a change of sign in the final result.

We say that the flux of fluid across the surface S is $\frac{1}{9}$. If the fluid velocity is in metres per second, and the coordinates along the x - and y -axes represent metres, then S has area 1 m^2 , and $\frac{1}{9} \text{ m}^3 \text{ s}^{-1}$ of fluid passes through S . ■

More generally, consider the flux of an arbitrary vector field $\mathbf{F}$ across the surface S of a small rectangular box with sides of length δx , δy and δz , as shown in Figure 2.26.

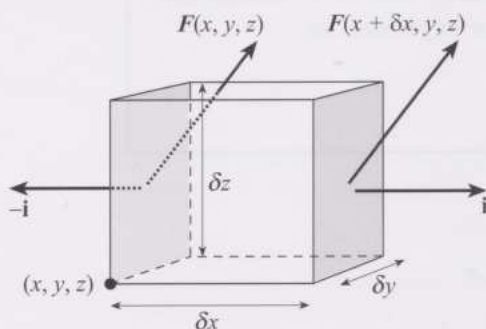


Figure 2.26 An elemental rectangular box

The flux across the right-hand shaded area of this box is approximately $\mathbf{F}(x + \delta x, y, z) \cdot \mathbf{i} \delta y \delta z$, whereas the flux across the left-hand shaded area is approximately $-\mathbf{F}(x, y, z) \cdot \mathbf{i} \delta y \delta z$ (because here the outward normal is $-\mathbf{i}$).

Applying a similar argument to the other two pairs of parallel surfaces, we obtain an approximation for the flux across the entire surface S :

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{A} &\simeq (\mathbf{F}(x + \delta x, y, z) - \mathbf{F}(x, y, z)) \cdot \mathbf{i} \delta y \delta z \\ &\quad + (\mathbf{F}(x, y + \delta y, z) - \mathbf{F}(x, y, z)) \cdot \mathbf{j} \delta x \delta z \\ &\quad + (\mathbf{F}(x, y, z + \delta z) - \mathbf{F}(x, y, z)) \cdot \mathbf{k} \delta x \delta y \\ &= \delta V \left(\frac{F_1(x + \delta x, y, z) - F_1(x, y, z)}{\delta x} \right. \\ &\quad \left. + \frac{F_2(x, y + \delta y, z) - F_2(x, y, z)}{\delta y} \right. \\ &\quad \left. + \frac{F_3(x, y, z + \delta z) - F_3(x, y, z)}{\delta z} \right), \end{aligned}$$

where $\delta V = \delta x \delta y \delta z$. If we now take the limit as the volume shrinks to zero, we obtain

$$\lim_{\delta V \rightarrow 0} \frac{1}{\delta V} \int_S \mathbf{F} \cdot d\mathbf{A} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \mathbf{F}, \quad (2.6)$$

where ∇ is the vector differential operator discussed on page 85. This result has an interesting consequence when $\mathbf{F}$ describes the velocity of a fluid.

The integral $\int_S \mathbf{F} \cdot d\mathbf{A}$ represents the volume of fluid crossing the surface in one second (both into and out of the region). For an incompressible fluid, where the amount of fluid entering the region equals the amount exiting, we would expect this volume to be zero (unless there is a source of fluid inside the region or a sink allowing some fluid to escape). In other words, for $\mathbf{F}$ to represent the velocity of an incompressible fluid, we require $\nabla \cdot \mathbf{F} = 0$ throughout the region.

The Gauss divergence theorem

Equation (2.6) is valid for an infinitesimally small volume element δV . The Gauss divergence theorem generalises this result to a finite volume R .

Gauss divergence theorem

For any (sufficiently smooth) vector field $\mathbf{F}$, the volume integral of $\nabla \cdot \mathbf{F}$ over a region R is equal to the (outward) flux of $\mathbf{F}$ across the surface S enclosing R . This statement can be expressed as

$$\int_R \nabla \cdot \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{A}.$$

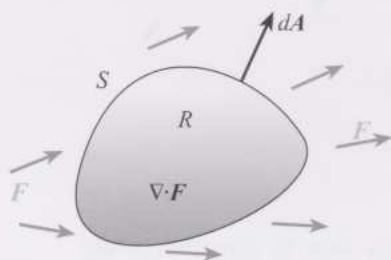


Figure 2.27 Schematic representation of the Gauss divergence theorem

This result is easily established from equation (2.6). First we use a Cartesian grid to divide the region R into a large number of box-shaped volumes δV_i with corresponding surfaces δS_i . Then

$$\int_R \nabla \cdot \mathbf{F} dV \simeq \sum_i (\nabla \cdot \mathbf{F})_i \delta V_i,$$

where $(\nabla \cdot \mathbf{F})_i$ means $\nabla \cdot \mathbf{F}$ evaluated at some point within the small volume δV_i . Using equation (2.6), we obtain

$$\int_R \nabla \cdot \mathbf{F} dV \simeq \sum_i \left(\int_{\delta S_i} \mathbf{F} \cdot d\mathbf{A}_i \right),$$

where $\int_{\delta S_i} \mathbf{F} \cdot d\mathbf{A}_i$ represents the flux of $\mathbf{F}$ out of the box-shaped region δV_i .

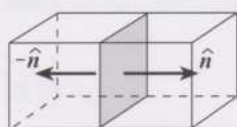


Figure 2.28 Two adjacent elemental boxes

Now, adjacent boxes have a rectangular face in common, and the outward normals on this common face have opposite signs (as shown in Figure 2.28). The flux of $\mathbf{F}$ out of this face from one box is equal in magnitude to the flux of $\mathbf{F}$ into the adjacent box, but these fluxes have opposite signs, so their contributions to the sum cancel. This cancellation occurs for all the faces except those at the boundary of the region R , so we have

$$\int_R \nabla \cdot \mathbf{F} dV \simeq \int_S \mathbf{F} \cdot d\mathbf{A}.$$

As the number of elements in the subdivision increases, and their size decreases, this approximation becomes more accurate, and in the limit we obtain the required result.

The Gauss divergence theorem is a very important result. It relates an integral over a region to an integral over the boundary of that region. In this sense it is a generalisation of the well-known result in one dimension

$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$. In fact, the theorem can be generalised to any number of dimensions. At a practical level, it enables us to calculate the flux of a vector field by calculating a volume integral of a scalar field. In general, this is much simpler than calculating integrals of vector fields over surfaces.

Here, the points a and b form the 'boundary' of the line interval $[a, b]$.

Example 2.25

Verify the Gauss divergence theorem for the vector field

$$\mathbf{F} = xy^2\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$$

and the region R given by the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

First we integrate $\nabla \cdot \mathbf{F}$ over the region R . Since

$$\nabla \cdot \mathbf{F} = \frac{\partial(xy^2)}{\partial x} + \frac{\partial(yz)}{\partial y} + \frac{\partial(zx)}{\partial z} = y^2 + z + x,$$

we have

$$\begin{aligned} \int_R \nabla \cdot \mathbf{F} dV &= \int_0^1 \left(\int_0^1 \left(\int_0^1 (x + y^2 + z) dx \right) dy \right) dz \\ &= \int_0^1 \left(\int_0^1 \left(\frac{1}{2} + y^2 + z \right) dy \right) dz \\ &= \int_0^1 \left(\frac{5}{6} + z \right) dz = \frac{4}{3}. \end{aligned}$$

Next we integrate the field $\mathbf{F}$ over the surface of R . Since R is a cube, we need to evaluate six integrals, one for each of the faces at $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

For the face at $x = 0$, the unit normal out of the cube is given by $\hat{\mathbf{n}} = -\mathbf{i}$, so the surface area element is $d\mathbf{A} = -\mathbf{i} dy dz$. Hence $\mathbf{F}(0, y, z) \cdot d\mathbf{A} = 0$, and we obtain

$$S_1 = \int_{x=0} \mathbf{F}(0, y, z) \cdot d\mathbf{A} = 0.$$

For the face at $x = 1$, the unit normal out of the cube is given by $\hat{\mathbf{n}} = \mathbf{i}$, so the surface area element is $d\mathbf{A} = \mathbf{i} dy dz$. Hence $\mathbf{F}(1, y, z) \cdot d\mathbf{A} = y^2 dy dz$, and we obtain

$$S_2 = \int_{x=1} \mathbf{F}(1, y, z) \cdot d\mathbf{A} = \int_0^1 \left(\int_0^1 y^2 dy \right) dz = \frac{1}{3}.$$

For the face at $y = 0$, we have $\hat{\mathbf{n}} = -\mathbf{j}$, so $d\mathbf{A} = -\mathbf{j} dx dz$. Hence we have $\mathbf{F}(x, 0, z) \cdot d\mathbf{A} = 0$, and we obtain

$$S_3 = \int_{y=0} \mathbf{F}(x, 0, z) \cdot d\mathbf{A} = 0.$$

For the face at $y = 1$, we have $d\mathbf{A} = \mathbf{j} dx dz$, so $\mathbf{F}(x, 1, z) \cdot d\mathbf{A} = z dx dz$, and we obtain

$$S_4 = \int_{y=1} \mathbf{F}(x, 1, z) \cdot d\mathbf{A} = \int_0^1 \left(\int_0^1 z dx \right) dz = \frac{1}{2}.$$

For the face at $z = 0$, we have $d\mathbf{A} = -\mathbf{k} dx dy$, so $\mathbf{F}(x, y, 0) \cdot d\mathbf{A} = 0$, and we obtain

$$S_5 = \int_{z=0} \mathbf{F}(x, y, 0) \cdot d\mathbf{A} = 0.$$

For the face at $z = 1$, we have $d\mathbf{A} = \mathbf{k} dx dy$, so $\mathbf{F}(x, y, 1) \cdot d\mathbf{A} = x dx dy$, and we obtain

$$S_6 = \int_{z=1} \mathbf{F}(x, y, 1) \cdot d\mathbf{A} = \int_0^1 \left(\int_0^1 x dx \right) dy = \frac{1}{2}.$$

Adding these surface integrals gives

$$S_1 + S_2 + S_3 + S_4 + S_5 + S_6 = \frac{4}{3},$$

the same result as obtained by finding the volume integral. ■

Exercise 2.23

Verify the Gauss divergence theorem for the vector field $\mathbf{F} = x \sin(\pi y)\mathbf{i} + xz\mathbf{j}$ over the region $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

2.4 Changing coordinates

In Subsection 2.2.8, we showed that if the Cartesian coordinates x and y are functions of two independent variables u and v , then any function $w(x, y)$ can be differentiated with respect to u and v , using the chain rule. First, keeping v fixed, we have

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}.$$

Now keeping u fixed, we have

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v}.$$

In this section we use these results to transform between derivatives in polar and Cartesian coordinates.

Polar to Cartesian coordinates

Recall that in polar coordinates, points (x, y) in the plane are labelled by (r, θ) where $x = r \cos \theta$ and $y = r \sin \theta$.

Now suppose that we have a function $w(x, y)$ and we wish to express its partial derivatives with respect to r and θ in terms of its partial derivatives with respect to x and y .

First we find

$$\frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta,$$

and then, using the chain rule,

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial w}{\partial x} \cos \theta + \frac{\partial w}{\partial y} \sin \theta \\ &= \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial w}{\partial x} + \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial w}{\partial y} \end{aligned}$$

and

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \theta} = -r \sin \theta \frac{\partial w}{\partial x} + r \cos \theta \frac{\partial w}{\partial y} = x \frac{\partial w}{\partial y} - y \frac{\partial w}{\partial x}.$$

Similarly, we can calculate the second-order partial derivatives with respect to r and θ .

The algebra may be unpleasant, but the process is straightforward.

Example 2.26

For an arbitrary function $w(x, y)$, calculate the second derivative with respect to θ in terms of the second derivatives with respect to x and y .

Solution

We have just shown that

$$\frac{\partial w}{\partial \theta} = x \frac{\partial w}{\partial y} - y \frac{\partial w}{\partial x};$$

since w is arbitrary, it is convenient to consider the operator

$$\frac{\partial}{\partial \theta} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$

This allows us to express $\frac{\partial^2 w}{\partial \theta^2}$ in terms of partial derivatives with respect to x and y in the form

$$\frac{\partial^2 w}{\partial \theta^2} = \frac{\partial}{\partial \theta} \left(\frac{\partial w}{\partial \theta} \right) = \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \left(x \frac{\partial w}{\partial y} - y \frac{\partial w}{\partial x} \right).$$

It remains to simplify the expression on the right:

$$\begin{aligned} \frac{\partial^2 w}{\partial \theta^2} &= \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \left(x \frac{\partial w}{\partial y} - y \frac{\partial w}{\partial x} \right) \\ &= x \frac{\partial}{\partial y} \left(x \frac{\partial w}{\partial y} \right) - x \frac{\partial}{\partial y} \left(y \frac{\partial w}{\partial x} \right) - y \frac{\partial}{\partial x} \left(x \frac{\partial w}{\partial y} \right) + y \frac{\partial}{\partial x} \left(y \frac{\partial w}{\partial x} \right) \\ &= x^2 \frac{\partial^2 w}{\partial y^2} - x \left(\frac{\partial w}{\partial x} + y \frac{\partial^2 w}{\partial x \partial y} \right) - y \left(\frac{\partial w}{\partial y} + x \frac{\partial^2 w}{\partial x \partial y} \right) + y^2 \frac{\partial^2 w}{\partial x^2} \\ &= x^2 \frac{\partial^2 w}{\partial y^2} - 2xy \frac{\partial^2 w}{\partial x \partial y} + y^2 \frac{\partial^2 w}{\partial x^2} - x \frac{\partial w}{\partial x} - y \frac{\partial w}{\partial y}. \quad \blacksquare \end{aligned}$$

Example 2.27

Given $z = x^2 + y^2$, calculate $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$ and $\frac{\partial^2 z}{\partial r \partial \theta}$ in terms of r and θ .

Solution

We have

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = 2x \cos \theta + 2y \sin \theta \\ &= 2r(\cos^2 \theta + \sin^2 \theta) = 2r, \\ \frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = 2x(-r \sin \theta) + 2y(r \cos \theta) \\ &= 2r^2(-\cos \theta \sin \theta + \sin \theta \cos \theta) = 0, \\ \frac{\partial^2 z}{\partial r \partial \theta} &= 0. \end{aligned}$$

In this example we could have noticed that $z = x^2 + y^2 = r^2$, from which the results follow immediately. $\blacksquare$

Cartesian to polar coordinates

Again we have the relationships $x = r \cos \theta$ and $y = r \sin \theta$, but while previously we considered x and y to be functions of r and θ , now we suppose that r and θ are functions of x and y .

Suppose that we wish to express $\frac{\partial^2 w}{\partial x^2}$ in terms of partial derivatives with respect to r and θ . One might suspect that it would be useful to write

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}(y/x) \quad \text{for } x > 0,$$

and proceed as before. However, using this approach the algebra becomes unpleasant, and there is a better way.

We begin with the equations $x = r \cos \theta$ and $y = r \sin \theta$. First we differentiate them partially with respect to x , to obtain

$$1 = \frac{\partial r}{\partial x} \cos \theta - r \sin \theta \frac{\partial \theta}{\partial x} \quad \text{and} \quad 0 = \frac{\partial r}{\partial x} \sin \theta + r \cos \theta \frac{\partial \theta}{\partial x},$$

and these equations can be solved to give

$$\frac{\partial r}{\partial x} = \cos \theta \quad \text{and} \quad \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}.$$

Note that $\frac{\partial r}{\partial x}$ and $\frac{\partial \theta}{\partial x}$ are not simply the inverses of $\frac{\partial x}{\partial r}$ and $\frac{\partial x}{\partial \theta}$.

Similarly, differentiating the equations $x = r \cos \theta$ and $y = r \sin \theta$ partially with respect to y , and solving the resulting equations, we obtain

$$\frac{\partial r}{\partial y} = \sin \theta \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}.$$

These relationships are exactly what we need in order to find expressions for $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in terms of $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$. From the chain rule, we have

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta} \frac{\partial \theta}{\partial x} = \cos \theta \frac{\partial w}{\partial r} - \frac{\sin \theta}{r} \frac{\partial w}{\partial \theta}$$

and

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta} \frac{\partial \theta}{\partial y} = \sin \theta \frac{\partial w}{\partial r} + \frac{\cos \theta}{r} \frac{\partial w}{\partial \theta}.$$

If we now replace w by $\frac{\partial w}{\partial x}$ in the first of these expressions, we obtain

$$\frac{\partial^2 w}{\partial x^2} = \cos \theta \frac{\partial}{\partial r} \left(\frac{\partial w}{\partial x} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial w}{\partial x} \right),$$

and using the above expression for $\frac{\partial w}{\partial x}$, this becomes

$$\frac{\partial^2 w}{\partial x^2} = \cos \theta \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial w}{\partial r} - \frac{\sin \theta}{r} \frac{\partial w}{\partial \theta} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial w}{\partial r} - \frac{\sin \theta}{r} \frac{\partial w}{\partial \theta} \right).$$

Finally, using the rule for differentiating a product, we have

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \cos^2 \theta \frac{\partial^2 w}{\partial r^2} - \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 w}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 w}{\partial \theta^2} \\ &\quad + \frac{\sin^2 \theta}{r} \frac{\partial w}{\partial r} + \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial w}{\partial \theta}. \end{aligned} \quad (2.7)$$

In the following example and exercise we assume that a function $z(r, \theta)$ becomes a function $z(x, y)$ after a change from polar to Cartesian coordinates. Thus the independent variables change from r and θ to x and y , while the dependent variable remains as z .

Example 2.28

Given $z = r \sin 2\theta$, show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$.

Solution

We have

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial x} = \sin 2\theta \cos \theta + 2r \cos 2\theta \left(-\frac{\sin \theta}{r} \right) \\ &= \sin 2\theta \cos \theta - 2 \cos 2\theta \sin \theta\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial y} = \sin 2\theta \sin \theta + 2r \cos 2\theta \frac{\cos \theta}{r} \\ &= \sin 2\theta \sin \theta + 2 \cos 2\theta \cos \theta,\end{aligned}$$

therefore

$$\begin{aligned}x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= r \cos \theta (\sin 2\theta \cos \theta - 2 \cos 2\theta \sin \theta) \\ &\quad + r \sin \theta (\sin 2\theta \sin \theta + 2 \cos 2\theta \cos \theta) \\ &= r \sin 2\theta (\cos^2 \theta + \sin^2 \theta) = z. \quad \blacksquare\end{aligned}$$

Exercise 2.24

Given a function $w(x, y)$, find an expression for $\frac{\partial^2 w}{\partial y^2}$ in terms of r and θ , where $x = r \cos \theta$ and $y = r \sin \theta$. Hence find an expression for $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ in terms of partial derivatives with respect to r and θ .

Laplace's equation in polar coordinates

The operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is known as the *Laplacian operator*; you will meet it again in Block II.

The partial differential equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

is the three-dimensional Cartesian form of **Laplace's equation**, which arises in many areas of applied mathematics. For various reasons, it is often useful to transform this equation into spherical or cylindrical polar coordinates, and to do that we require methods similar to those developed above. Applying the results of Exercise 2.24, the two-dimensional form of

Laplace's equation $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$ becomes

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial r} = 0.$$

(This form of Laplace's equation may appear to be more complicated than the original. In fact, as we shall see later when we discuss separation of variables in Block I, this form can be advantageous.)

Exercise 2.25

Transform the partial differential equation

$$a \frac{\partial w}{\partial x} + b \frac{\partial w}{\partial y} = 0,$$

where a and b are constants, by making the change of coordinates $u = bx - ay$ and $v = ax + by$, so the new independent variables are u and v .

Exercise 2.26

Transform the partial differential equation

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 0$$

by making the change of coordinates $u = x + y$ and $v = x - y$, so the new independent variables are u and v .

End-of-section Exercises**Exercise 2.27**

Find the cosine of the angle θ between the vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Exercise 2.28

Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, calculate

$$(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Exercise 2.29

- (a) Show that the curve $y^2 = 4ax$, where a is a constant, can be defined in parametric form by writing $x = at^2$ and $y = 2at$. Find the length of the curve from the origin to the point $(a, 2a)$. [Hint: Express the length as an integral in terms of t , then use integration by parts.]
- (b) Show that the surface

$$z = \frac{x^2}{4} - \frac{y^2}{9}$$

can be specified in terms of two parameters u and v in the form $x = 2(u + v)$, $y = 3(u - v)$, $z = 4uv$, and show that the point P with coordinates $x = 2$, $y = 3$, $z = 0$ lies on the surface and corresponds to choosing $u = 1$ and $v = 0$. Find the equation of the tangent plane to the surface at P , and find a vector perpendicular to this plane.

Exercise 2.30

- (a) Find the equation of the tangent plane to the surface

$$z = f(x, y) = (x + 2y + 1)(x - y + 2)$$

at the point $(0, 0, 2)$ on the surface. Show that the point corresponding to a vector of the form

$$\mathbf{r} = 2\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{j} + 3\mathbf{k}),$$

where λ and μ are arbitrary scalars, lies on this tangent plane.

- (b) Given $\phi(x, y, z) = z - f(x, y)$, evaluate $\nabla\phi$ at the point $(0, 0, 2)$. Show that any vector $\mathbf{r} - 2\mathbf{k}$ (which is parallel to the tangent plane at $(0, 0, 2)$) is perpendicular to $\nabla\phi$ evaluated at the point $(0, 0, 2)$.

Exercise 2.31

- (a) Calculate the gradient of the scalar field $F(x, y, z) = x^2 - 3xy + y^3 + 2xz + z$ at the point $(1, -1, 2)$.
- (b) Calculate the divergence of the vector field $\mathbf{F} = \sinh(xy)\mathbf{i} + \cosh(yz)\mathbf{j} + e^{zx}\mathbf{k}$.

Exercise 2.32

Find the work done by the force $\mathbf{F} = (x + yz)\mathbf{i} + (y + xz)\mathbf{j} + (z + xy)\mathbf{k}$ in moving a particle from the origin to the point P with coordinates $(1, 1, 1)$:

- (a) along the straight line joining the origin to P ;
- (b) along the curve $x = t$, $y = t^2$, $z = t^3$.

Exercise 2.33

- (a) The function ϕ is defined by $\phi(x, y, z) = x^2yz$, and the vector field $\mathbf{F}$ is given by $\mathbf{F} = \nabla\phi$. A point P with coordinates $(\cos t, \sin t, \sin 2t)$, $0 \leq t \leq 2\pi$, is an arbitrary point on a curve C . Show that the point P_1 corresponding to $t = 0$ coincides with the point P_2 corresponding to $t = 2\pi$, and show that

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0.$$

[Hint: Calculate $\frac{d}{dt}(\cos^2 t \sin t \sin 2t)$.]

- (b) Generalise the result of part (a) by considering an arbitrary vector field $\mathbf{F} = \nabla\phi$ for some function $\phi(x, y, z)$, where P , with coordinates $(x(t), y(t), z(t))$, $t_1 \leq t \leq t_2$, is an arbitrary point on a curve C . If the point P_1 corresponding to t_1 coincides with the point P_2 corresponding to t_2 , show that

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0.$$

You may assume that the curve C is smooth.

2.5 Outcomes

After studying this chapter you should be able to:

- understand the meaning of continuity for functions of many variables;
- understand the meaning of and be able to calculate partial derivatives;
- determine the tangent plane to a surface at a point;
- calculate first- and second-order Taylor approximations for functions of several variables;
- calculate and classify stationary points of functions of two variables;
- apply the chain rule for functions of several variables;
- integrate over two-dimensional planar regions in Cartesian and polar coordinates;
- integrate over non-planar regions when given an appropriate coordinate system;
- understand the concepts of scalar and vector fields;
- calculate the gradient of a scalar field and the divergence of a vector field;
- calculate the length of a curve specified either by a function or in parametric form;
- calculate line integrals of vector fields;
- calculate the integral of a vector field over a surface to determine the flux;
- verify the Gauss divergence theorem in elementary cases;
- use the chain rule to find derivatives with respect to new coordinates;
- appreciate the derivation of Laplace's equation in polar coordinates.

Solutions to Exercises in Chapter 2

Solution 2.1

The equation $y = x^2$ is recognisable as the equation of a parabola with the y -axis as its axis of symmetry. The equation $x = y^2$ also represents a parabola, but with the x -axis as its axis of symmetry. The equation $x = a + y^2$ represents a translation of the latter parabola along the x -axis. These observations allow us to sketch the following curves.

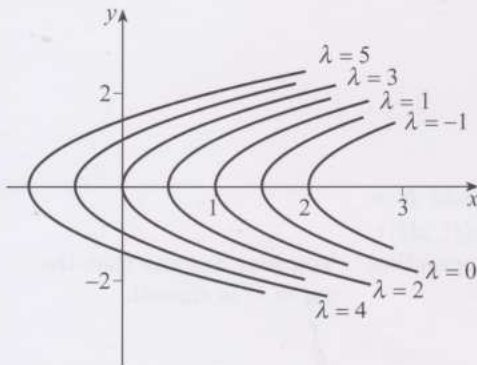


Figure 2.29 The curves $3 - 2x + y^2 = \lambda$

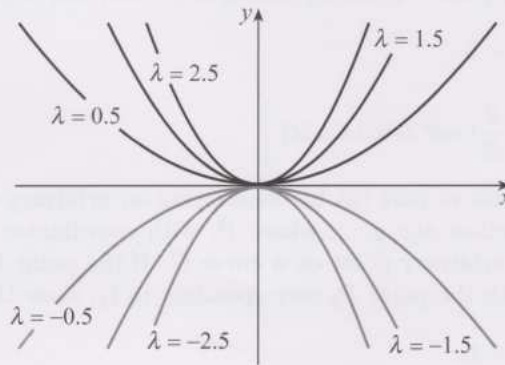


Figure 2.30 The curves $y/x^2 = \lambda$

The level curves in the first diagram are typical of a 'well-behaved' function, while the fact that the curves meet at the origin in the second figure means that the function is not defined there.

Solution 2.2

$$\frac{\partial F}{\partial x} = 2xy^3 + e^y \cos(xe^y) \text{ and } \frac{\partial F}{\partial y} = 3x^2y^2 + xe^y \cos(xe^y).$$

Solution 2.3

We have

$$\int_0^1 \sin(ax) \, dx = \left[-\frac{\cos(ax)}{a} \right]_0^1 = \frac{1}{a}(1 - \cos a),$$

and differentiating partially with respect to a gives

$$\frac{d}{da} \left(\int_0^1 \sin(ax) \, dx \right) = \frac{d}{da} \left(\frac{1}{a}(1 - \cos a) \right) = \frac{\sin a}{a} - \frac{1 - \cos a}{a^2}.$$

But, using Leibniz's rule,

$$\frac{d}{da} \left(\int_0^1 \sin(ax) \, dx \right) = \int_0^1 \frac{\partial}{\partial a} (\sin(ax)) \, dx = \int_0^1 x \cos(ax) \, dx,$$

therefore

$$\int_0^1 x \cos(ax) \, dx = \frac{\sin a}{a} - \frac{1 - \cos a}{a^2}.$$

Solution 2.4

$$(a) \quad \frac{\partial F}{\partial x} = ye^{xy} + 2x \text{ and } \frac{\partial F}{\partial y} = xe^{xy} + \cos y.$$

- (b) Differentiating partially with respect to x gives $\frac{1}{2}x^{-1/2} + \frac{1}{2}z^{-1/2}\frac{\partial z}{\partial x} = 0$, so $\frac{\partial z}{\partial x} = -\frac{\sqrt{z}}{\sqrt{x}}$ ($x > 0$). Similarly, $\frac{\partial z}{\partial y} = -\frac{\sqrt{z}}{\sqrt{y}}$ ($y > 0$).
- (c) If $\cos(xyz) = 1$, then $xyz = 2k\pi$ for some integer k , so $z = 2k\pi/(xy)$. It follows that $\frac{\partial z}{\partial x} = -2k\pi/(x^2y)$ and $\frac{\partial z}{\partial y} = -2k\pi/(xy^2)$.
- (d) (i) Putting $x = 1$ and $y = 2$ gives $z = -1$, thus the point $(1, 2, -1)$ does lie on the surface. The partial derivatives are

$$\frac{\partial z}{\partial x} = (y - 2x + 1) - 2(x - 2y + 1) = 5y - 4x - 1$$

and

$$\frac{\partial z}{\partial y} = 5x - 4y - 1.$$

Evaluating the partial derivatives when $x = 1$ and $y = 2$, we obtain $F_x(1, 2) = 5$ and $F_y(1, 2) = -4$. Thus the equation of the tangent plane is

$$z = 5(x - 1) - 4(y - 2) - 1.$$

- (ii) Putting $x = 1$ and $y = 1$ gives $z = 1$, so the point $(1, 1, 1)$ does lie on the surface. At this point, the partial derivatives are $F_x(1, 1) = F_y(1, 1) = 0$, from which we deduce that this is a stationary point. At $(1, 1)$, the equation of the tangent plane reduces to $z = 1$, so it is parallel to the (x, y) -plane.

Putting $s = x - 1$ and $t = y - 1$, we obtain

$$\begin{aligned} F(x, y) - F(1, 1) &= (s - 2t)(t - 2s) \\ &= -2s^2 + 5st - 2t^2 \\ &= -2(x - 1)^2 + 5(x - 1)(y - 1) - 2(y - 1)^2. \end{aligned}$$

You may have noticed that this expression contains no linear terms in powers of $x - 1$ and $y - 1$; this is a consequence of the fact that there is a stationary point at $(1, 1)$.

- (iii) Putting $y = 2 - x$ gives $F(x, y) - F(1, 1) = -9(x - 1)^2 \leq 0$, and putting $y = x$ gives $F(x, y) - F(1, 1) = (x - 1)^2 \geq 0$.

We conclude that every neighbourhood of the point $(1, 1)$ contains points at which $F(x, y) - F(1, 1)$ is positive and points at which it is negative. Thus the stationary point at $(1, 1)$ cannot be an extremum.

- (e) We have

$$\frac{\partial F}{\partial x} = (x^2 + y^2)e^x + 2xe^x = ((x + 1)^2 + y^2 - 1)e^x \quad \text{and} \quad \frac{\partial F}{\partial y} = 2ye^x,$$

so stationary points occur at the intersections of the circle $(x + 1)^2 + y^2 = 1$ with the line $y = 0$, namely at $(0, 0)$ and $(-2, 0)$.

Since $F(x, y) - F(0, 0) = (x^2 + y^2)e^x \geq 0$, the stationary point at the origin is a local minimum.

Shortly we shall see a simple criterion that will allow us to reach this conclusion more easily.

You can find these by substituting $y = 0$ into the equation $(x + 1)^2 + y^2 = 1$ to obtain $x = 0$ or $x = -2$.

Solution 2.5

- (a) $w_x = ye^{xy}$ and $w_y = xe^{xy}$, so $w_{xx} = y^2e^{xy}$, $w_{xy} = e^{xy} + xye^{xy}$ and $w_{yy} = x^2e^{xy}$. The quadratic approximation is therefore

$$\begin{aligned} w(x, y) &\simeq w(1, 2) + w_x(1, 2)(x - 1) + w_y(1, 2)(y - 2) \\ &\quad + \frac{1}{2}(w_{xx}(1, 2)(x - 1)^2 + 2w_{xy}(1, 2)(x - 1)(y - 2) \\ &\quad + w_{yy}(1, 2)(y - 2)^2) \\ &= e^2(1 + 2(x - 1) + (y - 2) + 2(x - 1)^2 + 3(x - 1)(y - 2) \\ &\quad + \frac{1}{2}(y - 2)^2). \end{aligned}$$

- (b) Since $\frac{\partial w}{\partial x} = \cos x \cosh y$ and $\frac{\partial w}{\partial y} = \sin x \sinh y$, the stationary points occur at the solutions of the pair of equations $\cos x \cosh y = 0$ and $\sin x \sinh y = 0$. Since $\cosh y > 1$, the first of these equations gives $\cos x = 0$, so

$$x = \frac{\pi}{2} + n\pi = \left(\frac{2n+1}{2}\right)\pi \quad \text{for } n \text{ an integer.}$$

For these values of x we know that $|\sin x| = 1$, so the second equation gives $y = 0$.

The second partial derivatives are

$$\frac{\partial^2 w}{\partial x^2} = -\sin x \cosh y, \quad \frac{\partial^2 w}{\partial x \partial y} = \cos x \sinh y, \quad \frac{\partial^2 w}{\partial y^2} = \sin x \cosh y.$$

Using the previous notation, at the point $(\frac{\pi}{2} + n\pi, 0)$ we have

$$PR - Q^2 = -(\sin x \cosh y)^2 - (\cos x \sinh y)^2 = -1,$$

so the stationary points are not extrema (but saddle points).

Solution 2.6

If $F(x, y) = f(x) + g(y)$, then $\frac{\partial F}{\partial y} = g'(y)$, so $\frac{\partial^2 F}{\partial x \partial y} = 0$.

(This simple observation will be of interest later, when we discuss the general solution of partial differential equations such as $F_{xy} = 0$.)

Solution 2.7

Putting $\mathbf{x} = \mathbf{0}$ (i.e. $x_1 = x_2 = x_3 = 0$), we obtain $A = F(\mathbf{0})$. Then differentiating partially with respect to each of the variables in turn, and putting $\mathbf{x} = \mathbf{0}$, we have

$$B_1 = \left(\frac{\partial F}{\partial x_1}\right)_{\mathbf{x}=\mathbf{0}}, \quad B_2 = \left(\frac{\partial F}{\partial x_2}\right)_{\mathbf{x}=\mathbf{0}}, \quad B_3 = \left(\frac{\partial F}{\partial x_3}\right)_{\mathbf{x}=\mathbf{0}}.$$

In order to find the remaining coefficients, we differentiate partially twice and then put $\mathbf{x} = \mathbf{0}$, giving

$$C_{11} = \frac{1}{2} \left(\frac{\partial^2 F}{\partial x_1^2}\right)_{\mathbf{x}=\mathbf{0}}, \quad C_{22} = \frac{1}{2} \left(\frac{\partial^2 F}{\partial x_2^2}\right)_{\mathbf{x}=\mathbf{0}}, \quad C_{33} = \frac{1}{2} \left(\frac{\partial^2 F}{\partial x_3^2}\right)_{\mathbf{x}=\mathbf{0}},$$

$$C_{12} = \left(\frac{\partial^2 F}{\partial x_1 \partial x_2}\right)_{\mathbf{x}=\mathbf{0}}, \quad C_{23} = \left(\frac{\partial^2 F}{\partial x_2 \partial x_3}\right)_{\mathbf{x}=\mathbf{0}}, \quad C_{31} = \left(\frac{\partial^2 F}{\partial x_3 \partial x_1}\right)_{\mathbf{x}=\mathbf{0}}.$$

Solution 2.8

- (a) We have $w = e^{\sin t \cos t}$, so

$$\frac{dw}{dt} = (\cos^2 t - \sin^2 t)e^{\sin t \cos t}.$$

Using the form $w = e^{xy}$, we have

$$\frac{\partial w}{\partial x} = ye^{xy}, \quad \frac{\partial w}{\partial y} = xe^{xy}, \quad \frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = -\sin t.$$

Hence, using the chain rule,

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = ye^{xy} \cos t - xe^{xy} \sin t \\ &= (\cos^2 t - \sin^2 t)e^{\sin t \cos t}, \end{aligned}$$

which agrees with our first calculation.

- (b) We have

$$\frac{\partial x}{\partial u} = a, \quad \frac{\partial x}{\partial v} = b, \quad \frac{\partial y}{\partial u} = c, \quad \frac{\partial y}{\partial v} = d.$$

Then, using the chain rule,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} = a \frac{\partial w}{\partial x} + c \frac{\partial w}{\partial y}$$

and

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} = b \frac{\partial w}{\partial x} + d \frac{\partial w}{\partial y}.$$

These relationships are valid for any function w , and it is convenient to write

$$\frac{\partial}{\partial u} = a \frac{\partial}{\partial x} + c \frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial v} = b \frac{\partial}{\partial x} + d \frac{\partial}{\partial y}.$$

Then

$$\frac{\partial^2 w}{\partial u^2} = \left(a \frac{\partial}{\partial x} + c \frac{\partial}{\partial y} \right) \left(a \frac{\partial w}{\partial x} + c \frac{\partial w}{\partial y} \right) = a^2 \frac{\partial^2 w}{\partial x^2} + 2ac \frac{\partial^2 w}{\partial x \partial y} + c^2 \frac{\partial^2 w}{\partial y^2},$$

$$\frac{\partial^2 w}{\partial u \partial v} = \left(a \frac{\partial}{\partial x} + c \frac{\partial}{\partial y} \right) \left(b \frac{\partial w}{\partial x} + d \frac{\partial w}{\partial y} \right) = ab \frac{\partial^2 w}{\partial x^2} + (bc + ad) \frac{\partial^2 w}{\partial x \partial y} + cd \frac{\partial^2 w}{\partial y^2},$$

$$\frac{\partial^2 w}{\partial v^2} = \left(b \frac{\partial}{\partial x} + d \frac{\partial}{\partial y} \right) \left(b \frac{\partial w}{\partial x} + d \frac{\partial w}{\partial y} \right) = b^2 \frac{\partial^2 w}{\partial x^2} + 2bd \frac{\partial^2 w}{\partial x \partial y} + d^2 \frac{\partial^2 w}{\partial y^2}.$$

Solution 2.9

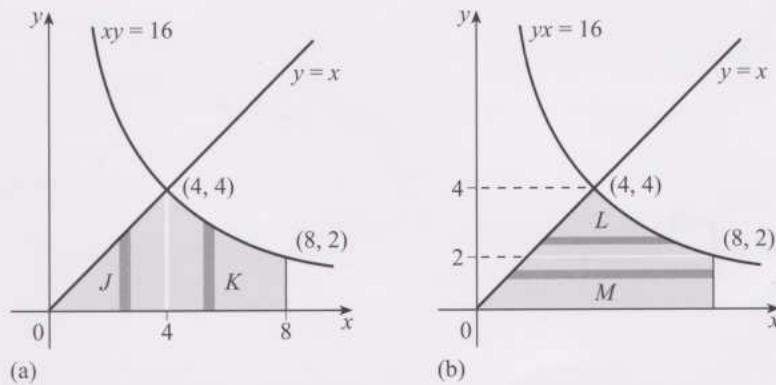


Figure 2.31 The region S divided into (a) vertical strips and (b) horizontal strips

There are two alternative approaches. We could divide the region S into two regions, J and K , as shown in Figure 2.31(a), where:

J is bounded by $y = 0$, $x = 4$ and $y = x$;

K is bounded by $x = 4$, $y = 0$, $x = 8$ and $y = 16/x$.

We then have

$$\int_S x^2 dA = \int_J x^2 dA + \int_K x^2 dA,$$

where the areas J and K written as repeated integrals are

$$\int_J x^2 dA = \int_0^4 x^2 \left(\int_0^x dy \right) dx = \int_0^4 x^3 dx = \left[\frac{1}{4} x^4 \right]_0^4 = 64$$

and

$$\int_K x^2 dA = \int_4^8 x^2 \left(\int_0^{16/x} dy \right) dx = \int_4^8 16x dx = [8x^2]_4^8 = 384.$$

Thus

$$\int_S x^2 dA = 64 + 384 = 448.$$

Alternatively, we could divide the region S into two regions L and M , as shown in Figure 2.31(b), where:

L is bounded by $y = 2$, $y = 16/x$ and $y = x$;

M is bounded by $y = 0$, $x = 8$, $y = 2$ and $y = x$.

We then have

$$\int_S x^2 dA = \int_L x^2 dA + \int_M x^2 dA,$$

where the areas L and M written as repeated integrals are

$$\begin{aligned} \int_L x^2 dA &= \int_2^4 \left(\int_y^{16/y} x^2 dx \right) dy = \int_2^4 \left[\frac{1}{3} x^3 \right]_y^{16/y} dy \\ &= \int_2^4 \left(\frac{4096}{3y^3} - \frac{y^3}{3} \right) dy = 108 \end{aligned}$$

and

$$\begin{aligned} \int_M x^2 dA &= \int_0^2 \left(\int_y^8 x^2 dx \right) dy = \int_0^2 \left[\frac{1}{3} x^3 \right]_y^8 dy \\ &= \int_0^2 \left(\frac{512}{3} - \frac{y^3}{3} \right) dy = 340. \end{aligned}$$

Thus

$$\int_S x^2 dA = 108 + 340 = 448,$$

as above.

Solution 2.10

We have

$$\int_S r^2 dr d\theta = \int_0^{\pi/2} \left(\int_0^1 r^2 dr \right) d\theta = \frac{1}{3} \int_0^{\pi/2} d\theta = \pi/6.$$

Solution 2.11

We have

$$\frac{\partial w}{\partial x} = -\frac{x}{(x^2 + y^2)^{3/2}} \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2} = \frac{2x^2 - y^2}{(x^2 + y^2)^{5/2}}.$$

Similarly (and taking advantage of the symmetry),

$$\frac{\partial^2 w}{\partial y^2} = \frac{2y^2 - x^2}{(x^2 + y^2)^{5/2}}.$$

Solution 2.12

We have

$$\frac{\partial F}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad \text{and} \quad \frac{\partial F}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2}.$$

Then

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{2y(3x^2 - y^2)}{(x^2 + y^2)^3}.$$

Solution 2.13

(a) If $w = e^x \cos y$, then $\frac{\partial w}{\partial x} = e^x \cos y$ and $\frac{\partial w}{\partial y} = -e^x \sin y$.

So $\frac{\partial^2 w}{\partial x^2} = e^x \cos y$ and $\frac{\partial^2 w}{\partial y^2} = -e^x \cos y$, which gives $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$.

(b) If $w = x^3 - 3xy^2$, then $\frac{\partial w}{\partial x} = 3x^2 - 3y^2$ and $\frac{\partial w}{\partial y} = -6xy$.

So $\frac{\partial^2 w}{\partial x^2} = 6x$ and $\frac{\partial^2 w}{\partial y^2} = -6x$, which gives $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$.

Solution 2.14

We have

$$F_x = -22 + 20x + 2y \quad \text{and} \quad F_y = -12 + 2x + 10y.$$

The single stationary point occurs at the intersection of the lines $10x + y = 11$ and $x + 5y = 6$, and thus is at $(1, 1)$. Since $F_{xx} = 20$, $F_{yy} = 10$ and $F_{xy} = 2$, it follows that (using the standard notation)

$$PR - Q^2 = 200 - 4 = 196 > 0,$$

therefore the stationary point is a local minimum.

Solution 2.15

Let P , with coordinates (x, y, z) , be an arbitrary point lying in the given plane. Then the distance from P to the origin is $d = \sqrt{x^2 + y^2 + z^2}$. The algebra is simpler if we minimise $w = d^2 = x^2 + y^2 + z^2$ rather than d , and we note that on the plane we have $y = 2x + 2z - 16$, so $w = x^2 + (2x + 2z - 16)^2 + z^2$. As P ranges over the plane, there will be a point which is closest to the origin, and at this point the function $w(x, z)$ will have a minimum. At this stationary point,

$$\frac{\partial w}{\partial x} = 2x + 4(2x + 2z - 16) = 10x + 8z - 64 = 0$$

and

$$\frac{\partial w}{\partial z} = 4(2x + 2z - 16) + 2z = 8x + 10z - 64 = 0,$$

therefore $x = z = 32/9$, so $y = -16/9$. Since there is only one stationary point, we know that this must be the required minimum, and it occurs at $(32/9, -16/9, 32/9)$.

We can confirm this conclusion by performing the evaluation

$$\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial z^2} - \left(\frac{\partial^2 w}{\partial x \partial z} \right)^2 = 10 \times 10 - 8^2 = 36 > 0,$$

so the stationary point is an extremum, and since $\frac{\partial^2 w}{\partial x^2} = 10 > 0$ this extremum is a minimum.

Note that here we have chosen to eliminate y , but we could equally well have chosen to eliminate x or z .

Solution 2.16

We have

$$\frac{\partial^3 w}{\partial x \partial y \partial z} = \frac{\partial^3 w}{\partial z \partial x \partial y} = \frac{\partial^3 w}{\partial y \partial z \partial x} = \frac{3xyz}{(x^2 + y^2 + z^2)^{5/2}}.$$

Solution 2.17

We have

$$\int_0^\pi \left(\int_0^{\cos \theta} \rho \sin \theta \, d\rho \right) d\theta = \int_0^\pi \left[\frac{1}{2} \rho^2 \sin \theta \right]_0^{\cos \theta} d\theta = \frac{1}{2} \int_0^\pi \cos^2 \theta \sin \theta \, d\theta.$$

Now we can either make the change of variable $u = \cos \theta$, or note that $\sin \theta \, d\theta = -d(\cos \theta)$:

$$\int_0^\pi \left(\int_0^{\cos \theta} \rho \sin \theta \, d\rho \right) d\theta = -\frac{1}{2} \int_0^\pi \cos^2 \theta \, d(\cos \theta) = \frac{1}{6} [-\cos^3 \theta]_0^\pi = \frac{1}{3}.$$

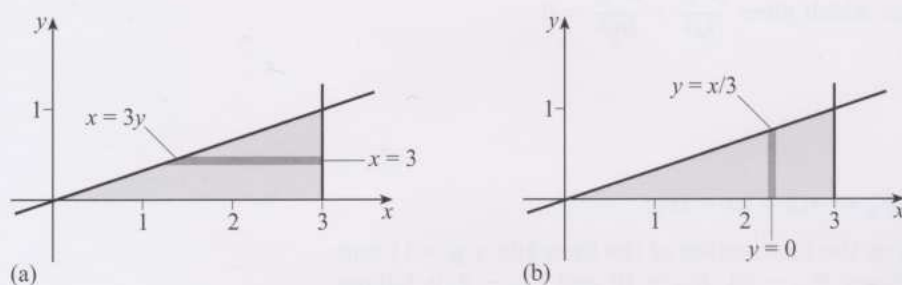
Solution 2.18

Figure 2.32

Dividing the region into horizontal strips, then summing over the strips (see Figure 2.32(a)), we obtain

$$\iint_S \exp(x^2) dA = \int_0^1 \left(\int_{3y}^3 \exp(x^2) dx \right) dy,$$

which cannot be evaluated directly in terms of the elementary functions.

Alternatively, dividing the region into vertical strips, then summing over the strips (see Figure 2.32(b)), we obtain

$$\begin{aligned} \iint_S \exp(x^2) dA &= \int_0^3 \left(\int_0^{x/3} \exp(x^2) dy \right) dx = \int_0^3 [y \exp(x^2)]_0^{x/3} dx \\ &= \int_0^3 \left[\frac{1}{3} x \exp(x^2) \right] dx \\ &= \frac{1}{6} [\exp(x^2)]_0^3 \\ &= \frac{1}{6} (e^9 - 1). \end{aligned}$$

The point to notice here is that the order in which we perform the integration may well affect the difficulty of the calculation.

Solution 2.19

(a) We have

$$|\mathbf{a}| = \sqrt{1+4+1} = \sqrt{6} \quad \text{and} \quad |\mathbf{b}| = \sqrt{4+1+1} = \sqrt{6},$$

and

$$\mathbf{a} \cdot \mathbf{b} = (1, 2, -1) \cdot (2, 1, 1) = 3.$$

So

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{1}{2}.$$

(b) The magnitude of the vector $\mathbf{c}$ is $\sqrt{3^2 + 2^2 + 5^2} = \sqrt{38}$.

Solution 2.20

We have

$$F_x = 2x \sin[(x+y)\pi] + (x^2 + 2y^2)\pi \cos[(x+y)\pi]$$

and

$$F_y = 4y \sin[(x+y)\pi] + (x^2 + 2y^2)\pi \cos[(x+y)\pi].$$

It follows that $F_x(1, 1) = F_y(1, 1) = 3\pi$, therefore $\nabla F = 3\pi(\mathbf{i} + \mathbf{j})$.

Solution 2.21

(a) We have

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = -\frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3}{(x^2 + y^2 + z^2)^{1/2}} \\ &= \frac{2}{(x^2 + y^2 + z^2)^{1/2}}.\end{aligned}$$

(b) We have

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k},$$

therefore $\nabla \cdot (\nabla f) = 6$.**Solution 2.22**(a) $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$, so the length of the curve is given by

$$\begin{aligned}\int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt &= \int_0^4 \sqrt{(2t)^2 + (3t^2)^2} dt \\ &= \int_0^4 t(4 + 9t^2)^{1/2} dt \\ &= \frac{1}{27} \left[(4 + 9t^2)^{3/2} \right]_0^4 \\ &= \frac{8}{27} (37\sqrt{37} - 1).\end{aligned}$$

(b) We have $x'(t) = 1$, $y'(t) = \sqrt{2}t$, $z'(t) = t^2$, so the distance travelled is

$$\begin{aligned}\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt &= \int_0^2 \sqrt{1 + 2t^2 + t^4} dt \\ &= \int_0^2 (1 + t^2) dt \\ &= \left[t + \frac{1}{3}t^3 \right]_0^2 = 4\frac{2}{3}.\end{aligned}$$

(c) The straight line joining the origin to the point $(1, 2, 3)$ can be parametrised as $x = t$, $y = 2t$, $z = 3t$ from $t = 0$ to $t = 1$. Then $\mathbf{r} = (t + 2\mathbf{j} + 3\mathbf{k})t$ and

$$\begin{aligned}\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_0^1 (2t^3\mathbf{i} + 3t\mathbf{j} + 4t^3\mathbf{k}) \cdot (t\mathbf{i} + 2t\mathbf{j} + 3t\mathbf{k}) dt \\ &= \int_0^1 (6t + 14t^3) dt = 6\frac{1}{2}.\end{aligned}$$

Solution 2.23First we integrate $\nabla \cdot \mathbf{F}$ over the region R . Since

$$\nabla \cdot \mathbf{F} = \frac{\partial(x \sin(\pi y))}{\partial x} + \frac{\partial(xz)}{\partial y} = \sin(\pi y),$$

we have

$$\begin{aligned}\int_R \nabla \cdot \mathbf{F} dV &= \int_0^1 \left(\int_0^1 \left(\int_0^1 \sin(\pi y) dx \right) dy \right) dz \\ &= \int_0^1 \left(\int_0^1 \sin(\pi y) dy \right) dz \\ &= \int_0^1 \frac{2}{\pi} dz = \frac{2}{\pi}.\end{aligned}$$

Next we integrate the field $\mathbf{F}$ over the surface of R . Since R is a cube, we need to evaluate six integrals, one for each of the faces at $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

For the face at $x = 0$, we have $d\mathbf{A} = -\mathbf{i} dy dz$, so $\mathbf{F}(0, y, z) \cdot d\mathbf{A} = 0$, and we obtain

$$S_1 = \int_{x=0} \mathbf{F}(0, y, z) \cdot d\mathbf{A} = 0.$$

For the face at $x = 1$, we have $d\mathbf{A} = \mathbf{i} dy dz$, so $\mathbf{F}(1, y, z) \cdot d\mathbf{A} = \sin(\pi y) dy dz$, and we obtain

$$S_2 = \int_{x=1} \mathbf{F}(1, y, z) \cdot d\mathbf{A} = \int_0^1 \left(\int_0^1 \sin(\pi y) dy \right) dz = \frac{2}{\pi}.$$

For the face at $y = 0$, we have $d\mathbf{A} = -\mathbf{j} dx dz$, so $\mathbf{F}(x, 0, z) \cdot d\mathbf{A} = -xz dx dz$, and we obtain

$$S_3 = \int_{y=0} \mathbf{F}(x, 0, z) \cdot d\mathbf{A} = - \int_0^1 \left(\int_0^1 xz dx \right) dz = -\frac{1}{4}.$$

For the face at $y = 1$, we have $d\mathbf{A} = \mathbf{j} dx dz$, so $\mathbf{F}(x, 1, z) \cdot d\mathbf{A} = xz dx dz$, and we obtain

$$S_4 = \int_{y=1} \mathbf{F}(x, 1, z) \cdot d\mathbf{A} = \int_0^1 \left(\int_0^1 xz dx \right) dz = \frac{1}{4}.$$

There is no contribution to the surface integral from the faces $z = 0$ and $z = 1$, since in each case $\mathbf{F} \cdot d\mathbf{A} = 0$.

Adding the surface integrals gives

$$S_1 + S_2 + S_3 + S_4 = \frac{2}{\pi},$$

the same result as obtained by finding the volume integral.

Solution 2.24

We have

$$\begin{aligned} \frac{\partial^2 w}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial y} \right) \\ &= \sin \theta \frac{\partial}{\partial r} \left(\sin \theta \frac{\partial w}{\partial r} + \frac{\cos \theta}{r} \frac{\partial w}{\partial \theta} \right) + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial w}{\partial r} + \frac{\cos \theta}{r} \frac{\partial w}{\partial \theta} \right) \\ &= \sin^2 \theta \frac{\partial^2 w}{\partial r^2} + \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 w}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{\cos^2 \theta}{r} \frac{\partial w}{\partial r} - \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial w}{\partial \theta}. \end{aligned}$$

Hence, using equation (2.7),

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial r}.$$

Solution 2.25

From the chain rule we have

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} = b \frac{\partial w}{\partial u} + a \frac{\partial w}{\partial v}$$

and

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y} = -a \frac{\partial w}{\partial u} + b \frac{\partial w}{\partial v},$$

therefore

$$a \frac{\partial w}{\partial x} + b \frac{\partial w}{\partial y} = a \left(b \frac{\partial w}{\partial u} + a \frac{\partial w}{\partial v} \right) + b \left(-a \frac{\partial w}{\partial u} + b \frac{\partial w}{\partial v} \right) = (a^2 + b^2) \frac{\partial w}{\partial v}.$$

The given partial differential equation therefore reduces to

$$\frac{\partial w}{\partial v} = 0.$$

Solution 2.26

We have

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1.$$

It follows from the chain rule that

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

and

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}.$$

These relations are valid for any function w , and it is convenient to write

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

and

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v},$$

in which case we have

$$\frac{\partial^2 w}{\partial x^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \right) = \frac{\partial^2 w}{\partial u^2} + 2 \frac{\partial^2 w}{\partial u \partial v} + \frac{\partial^2 w}{\partial v^2}$$

and

$$\frac{\partial^2 w}{\partial y^2} = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) \left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v} \right) = \frac{\partial^2 w}{\partial u^2} - 2 \frac{\partial^2 w}{\partial u \partial v} + \frac{\partial^2 w}{\partial v^2}.$$

It follows that

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 4 \frac{\partial^2 w}{\partial u \partial v},$$

so the given partial differential equation reduces to

$$\frac{\partial^2 w}{\partial u \partial v} = 0.$$

Solution 2.27

We have

$$\begin{aligned} \cos \theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k})}{\sqrt{2^2 + (-1)^2 + 3^2} \sqrt{1^2 + 2^2 + (-1)^2}} \\ &= \frac{2 - 2 - 3}{\sqrt{14}\sqrt{6}} \\ &= -\frac{3}{2\sqrt{21}}. \end{aligned}$$

Solution 2.28

We have

$$\mathbf{a} \cdot \mathbf{c} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 5$$

and

$$\mathbf{a} \cdot \mathbf{b} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 3,$$

so

$$\begin{aligned} (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} &= 5\mathbf{b} - 3\mathbf{c} = 5(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - 3(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &= 7\mathbf{i} + 13\mathbf{j} - 11\mathbf{k}. \end{aligned}$$

Solution 2.29

- (a) Since $t = \frac{y}{2a}$, we have $x = at^2 = a \left(\frac{y}{2a}\right)^2 = \frac{y^2}{4a}$, so $y^2 = 4ax$.

Since $x = at^2$ and $y = 2at$, we have

$$\frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a.$$

Thus the length of the curve is

$$S = \int_0^1 \sqrt{(2at)^2 + (2a)^2} dt = 2a \int_0^1 \sqrt{1+t^2} dt.$$

Putting $I = \int_0^1 \sqrt{1+t^2} dt$ and integrating by parts, we have

$$\begin{aligned} I &= \left[t\sqrt{1+t^2} \right]_0^1 - \int_0^1 \frac{t^2}{\sqrt{1+t^2}} dt \\ &= \sqrt{2} - \int_0^1 \frac{1+t^2}{\sqrt{1+t^2}} dt + \int_0^1 \frac{1}{\sqrt{1+t^2}} dt \\ &= \sqrt{2} - I + \int_0^1 \frac{1}{\sqrt{1+t^2}} dt \\ &= \sqrt{2} - I + [\sinh^{-1} t]_0^1, \end{aligned}$$

so

$$S = 2aI = a \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right).$$

- (b) Just as a curve can be parametrised by a single parameter as in part (a), so a surface can be parametrised by two parameters, and here we have $x = 2(u+v)$, $y = 3(u-v)$ and $z = 4uv$. Solving the first two of these equations gives

$$2u = \frac{x}{2} + \frac{y}{3} \quad \text{and} \quad 2v = \frac{x}{2} - \frac{y}{3}.$$

Then substituting into the third equation gives

$$z = \frac{x^2}{4} - \frac{y^2}{9},$$

showing that the parametrisation in terms of u , v and w does specify the given surface.

Putting $u = 1$ and $v = 0$, we obtain $x = 2$, $y = 3$ and $z = 0$, so the point $(2, 3, 0)$ does lie on the surface.

Differentiating $z = \frac{x^2}{4} - \frac{y^2}{9}$ partially gives

$$\frac{\partial z}{\partial x} = \frac{x}{2} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{2y}{9},$$

so $z_x(2, 3) = 1$ and $z_y(2, 3) = -\frac{2}{3}$. The equation of the tangent plane at the point $(2, 3, 0)$ is therefore

$$z = (x - 2) - \frac{2}{3}(y - 3),$$

which can be written as $3x - 2y - 3z = 0$.

There are several ways of finding a vector perpendicular to the tangent plane, but the following is probably the simplest. First we notice that the plane $3x - 2y - 3z = 0$ passes through the origin, and if we let (x, y, z) be any vector from the origin to a point in this plane, we can see that $(3, -2, -3) \cdot (x, y, z) = 3x - 2y - 3z = 0$. So the vectors (x, y, z) and $(3, -2, -3)$ are orthogonal, thus the vector $3\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ is perpendicular to the tangent plane at $(2, 3, 0)$.

It is also worth noting that the equation of the plane may be written as $f(x, y, z) = 3x - 2y - 3z = 0$, and then the vector $\nabla f = 3\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ is perpendicular to this plane. This is because ∇f is always perpendicular to the (tangent plane to the) level curves of f .

Solution 2.30

(a) We have

$$\frac{\partial f}{\partial x} = (x + 2y + 1) + (x - y + 2) = 2x + y + 3$$

and

$$\frac{\partial f}{\partial y} = 2(x - y + 2) - (x + 2y + 1) = x - 4y + 3.$$

The equation of the tangent plane at $(0, 0, 2)$ is therefore

$$z = f_x(0, 0)x + f_y(0, 0)y + f(0, 0) = 3x + 3y + 2.$$

The point corresponding to the vector $\mathbf{r} = 2\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{j} + 3\mathbf{k})$ has coordinates $(\lambda, \mu, 2 + 3\lambda + 3\mu)$, and these satisfy the equation $z = 3x + 3y + 2$, so the point does lie in the plane.

(b) We have

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k},$$

whose value at $(0, 0, 2)$ is $-3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

It follows that

$$\nabla\phi \cdot (\mathbf{r} - 2\mathbf{k}) = (-3\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (\lambda\mathbf{i} + \mu\mathbf{j} + 3(\lambda + \mu)\mathbf{k}) = 0,$$

so the vectors are perpendicular.

In order to clarify what is happening here, note that the level curve $\phi(x, y, z) = z - f(x, y) = 0$ corresponds to the surface $z = f(x, y)$. So at a given point on the surface, vectors which lie in the tangent plane to $z = f(x, y)$ will always be perpendicular to $\nabla\phi$.

Solution 2.31

(a) We have

$$F_x = 2x - 3y + 2z, \quad F_y = -3x + 3y^2, \quad F_z = 2x + 1,$$

so

$$\nabla F = (2x - 3y + 2z)\mathbf{i} + (-3x + 3y^2)\mathbf{j} + (2x + 1)\mathbf{k}.$$

At the point $(1, -1, 2)$, we obtain $\nabla F = 9\mathbf{i} + 3\mathbf{k}$.

(b) We have

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{\partial(\sinh(xy))}{\partial x} + \frac{\partial(\cosh(yz))}{\partial y} + \frac{\partial(e^{zx})}{\partial z} \\ &= y \cosh(xy) + z \sinh(yz) + xe^{zx}. \end{aligned}$$

Solution 2.32(a) We can parametrise the straight line joining the origin to the point P by writing $x = t$, $y = t$ and $z = t$ with $0 \leq t \leq 1$. Then

$$\begin{aligned} \mathbf{F}(\mathbf{r}) &= (x + yz)\mathbf{i} + (y + xz)\mathbf{j} + (z + xy)\mathbf{k} \\ &= (t + t^2)\mathbf{i} + (t + t^2)\mathbf{j} + (t + t^2)\mathbf{k} \\ &= (t + t^2)(\mathbf{i} + \mathbf{j} + \mathbf{k}) \end{aligned}$$

and

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k},$$

so the work done is

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_0^1 (t + t^2)(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) dt \\ &= 3 \int_0^1 (t + t^2) dt = 3 \left[\frac{1}{2}t^2 + \frac{1}{3}t^3 \right]_0^1 = \frac{5}{2}. \end{aligned}$$

- (b) The particle moves along the curve $x = t$, $y = t^2$, $z = t^3$ from the origin to the point P as t increases from 0 to 1. On this curve, we have

$$\mathbf{F}(\mathbf{r}) = (x + yz)\mathbf{i} + (y + xz)\mathbf{j} + (z + xy)\mathbf{k} = (t + t^5)\mathbf{i} + (t^2 + t^4)\mathbf{j} + 2t^3\mathbf{k}$$

and

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k},$$

so the work done is

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_0^1 ((t + t^5)\mathbf{i} + (t^2 + t^4)\mathbf{j} + 2t^3\mathbf{k}) \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) dt \\ &= \int_0^1 (t + t^5 + 2t(t^2 + t^4) + 6t^5) dt \\ &= \int_0^1 (t + 2t^3 + 9t^5) dt = \frac{5}{2}. \end{aligned}$$

Generally, the value of a line integral depends upon the path of integration; this is an example of one that does not. For the general vector field

$$\mathbf{F}(\mathbf{r}) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k},$$

it can be shown that the integral $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is independent of the path C between two given end points if there exists a function $\phi(x, y, z)$ such that

$$F_1 = \frac{\partial \phi}{\partial x}, \quad F_2 = \frac{\partial \phi}{\partial y}, \quad F_3 = \frac{\partial \phi}{\partial z};$$

in other words, if $\mathbf{F}(\mathbf{r}) = \nabla \phi$. In this example it is easy to verify that we can use $\phi = \frac{1}{2}(x^2 + y^2 + z^2) + xyz$.

Solution 2.33

- (a) We have

$$(\cos 0, \sin 0, \sin 0) = (1, 0, 0) = (\cos 2\pi, \sin 2\pi, \sin 4\pi),$$

so P_1 and P_2 are the same point.

Now

$$\mathbf{F} = \nabla \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k},$$

so

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_C (2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \int_0^{2\pi} (2x(t)y(t)z(t)\mathbf{i} + x(t)^2z(t)\mathbf{j} + x(t)^2y(t)\mathbf{k}) \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \right) dt \\ &= \int_0^{2\pi} (2\cos t \sin t \sin 2t \mathbf{i} + \cos^2 t \sin 2t \mathbf{j} + \cos^2 t \sin t \mathbf{k}) \cdot (-\sin t \mathbf{i} + \cos t \mathbf{j} + 2\cos 2t \mathbf{k}) dt \\ &= \int_0^{2\pi} (-2\cos t \sin^2 t \sin 2t + \cos^3 t \sin 2t + 2\cos^2 t \sin t \cos 2t) dt \\ &= \int_0^{2\pi} \frac{d}{dt} (\cos^2 t \sin t \sin 2t) dt \\ &= [\cos^2 t \sin t \sin 2t]_0^{2\pi} = 0. \end{aligned}$$

- (b) Using the chain rule, we have

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_C \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} \right) \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \right) dt \\ &= \int_{t_1}^{t_2} \left(\frac{\partial \phi}{\partial x} \frac{dx}{dt} + \frac{\partial \phi}{\partial y} \frac{dy}{dt} + \frac{\partial \phi}{\partial z} \frac{dz}{dt} \right) dt \\ &= \int_{t_1}^{t_2} \left(\frac{d\phi}{dt} \right) dt = [\phi(t)]_{t_1}^{t_2} = 0, \quad \text{since } \phi(t_1) = \phi(t_2). \end{aligned}$$

INDEX

- Argand diagram 22
- argument of a complex number 24
- auxiliary equation 37
- boundary conditions 39
- cardioid 76
- Cartesian components of a vector 81
- Cartesian form of a complex number 22
- chain rule 73
- codomain 5
- coefficients of a series 9
- complementary function 37
- complex exponential function 25
- complex function of a real variable 6
- complex numbers 21
- complex plane 22
- conjugate of a complex number 22
- continuity 59, 60
- continuous function 6
- contours 58
- convergence of a series 7
- convergent integral 16, 28
- convergent series 7
- cosine function 10, 25
- cylinder surface area 79
- cylindrical polar coordinates 79
- de Moivre's theorem 24
- definite integral 13
- degree of a polynomial 7
- del 85
- density 14, 82
- dependent variable 62
- derivative 6, 12
- derived function 6
- differentiable function 6
- differential equation 28
- divergence of a function 86
- divergence theorem 94
- divergent integral 16
- divergent series 7
- domain 5
- dot product 81, 82
- equating coefficients 19
- equating real and imaginary parts 22
- equation of a plane 64
- Euler's formula 25
- even function 5
- exponential function 10
- extrema 64, 68, 71
- factorising 20
- first-order partial derivative 61
- flux 91, 92
- function 5
 - of many variables 70
 - of two variables 58
- fundamental theorem of algebra 21
- fundamental theorem of calculus 13
- Gauss divergence theorem 94
- Gaussian function 12, 17
- Gaussian integral 17, 77
- general solution of a differential equation 31, 37
- geometric series 7
- gradient of a function 83
- homogeneous differential equation 31, 37
- homogeneous first-order differential equation 34
- hyperbolic functions 11, 12
- imaginary number 22
- imaginary part of a complex number 22
- indefinite integral 12
- independent variable 62
- infinite integral 15, 16
- infinite series 7
- initial conditions 39
- integral 12
 - infinite 15, 16
 - multiple 73, 78
 - over a non-planar surface 78
 - over a plane surface 73
 - repeated 74
 - triple 78
 - using polar coordinates 76
- integrand 13
- integrating factor 35
- integration 12, 73, 86
 - and differentiation 63
- inverse function 5
- Laplace's equation 80, 99
 - in polar coordinates 99
- Laplacian operator 99
- Leibniz's rule 63
- length of a curve 87
- length of a vector 81
- level curves 58, 102
- line integral 88–90
- linear approximation 63, 64
- linear density 14
- linear differential equation 31
- linear equation of first order 35
- linear equation with constant coefficients 37
- local extrema 64
- local maximum 64
- local minimum 64
- logarithm function 10
- magnitude of a vector 81
- maximum 64
- method of undetermined coefficients 37
- minimum 64
- mixed derivative theorem 66, 79
- modulus of a complex number 22, 24
- multiple integral 73, 78
- nabla 85

- natural logarithm 10
- non-linear differential equation 31, 54
- normal vector 91
- O notation 8
- odd function 5
- operator 6
- order notation 8
- order of a differential equation 31
- order of magnitude 8
- ordinary differential equation 29
- orthogonal functions 55
- parametric curve 87
- partial derivative 61, 66
 - first-order 61
 - higher-order 65
 - second-order 66
- partial differential equation 29
- partial fractions 18, 44
- partial sum 7
- particular integral 31, 37
- particular solution 32
- polar coordinates 24, 76, 77, 79, 96, 99
- polar form of a complex number 24
- polynomial 7
- primitive 12
- principal value 24
- projection 81
- quadratic approximation 66
- quadratic factor 20
- radius of convergence 9
- rate of convergence 8
- rational function 7, 18
- real function 5
- real line 5, 22
- real numbers 5, 22
- real part of a complex number 22
- remainder term 8
- repeated integral 74
- roots of a complex number 25
- saddle point 65
- scalar 80
- scalar field 82, 85
- scalar product 81, 82
- second-order partial derivative 66
- separable equation 32
- series 7
- sine function 10, 25
- slope of a surface 83, 84
- smooth curve 87
- smooth function 6, 61
- sphere
 - surface area 79
 - volume 14
- stationary point 64, 68, 71, 72
- sum of a series 7
- surface area
 - of a cylinder 79
 - of a sphere 79
- tangent function 11
- tangent plane 61, 63, 64
- Taylor approximation of order two 67, 70
- Taylor series 9, 10, 67
- transform 6
- triangle inequality 27
- trigonometric functions 10–12
- triple integral 78
- uniform wire 14
- unit vector 81, 91
- vector 80
- vector differential operator 85
- vector field 86
- volume integral 78
- volume of a sphere 14
- zero vector 81

MS324 Waves, diffusion and variational principles

Block 0

► Preparatory material

Block I

Waves

Block II

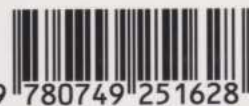
Random walks and diffusion

Block III

Variational principles



MS324 Block 0
ISBN 978 0 7492 5162 8



9 780749 251628